THE CONDENSER PRINCIPLE AND THE EFFECTIVE RESISTANCE ON A NETWORK

E. BENDITO, A. CARMONA AND A.M. ENCINAS

Depto. Matemática Aplicada III, UPC, Spain.
e-mail: angeles.carmona@upc.es

Notation used throughout the poster:

- $\Gamma=(V,E,c)$ is a network, with $|V|=n$ and $c(x)=\sum_{y\sim x} c(x,y)$
- $F\subset V$, $\delta(F)=\{x\in F^c:(x,y)\in E$ for some $y\in F^c\}$ vertex boundary
- $\partial(F)=\{(x,y)\in E: x\in F, y\in F^c\}$ edge boundary
- $Lu(x)=\sum_{y\sim x} c(x,y)(u(x)-u(y))$ is the Laplacian of Γ
- $I(u)=\langle Lu,u\rangle$ is the energy of u
- $\mu\in M(F)$ is a positive measure on F
- $S(\mu)=\{x\in V:\mu(x)\neq 0\}$ support of μ
- ν^F equilibrium measure for F
- ν_y equilibrium measure for $V-\{y\}$

POTENTIAL THEORY

$L: V\times V\to R$ is a symmetric kernel verifying:

- Energy principle: L is strictly positive definite on $\{\mu\in R^n:\langle\mu,1\rangle=0\}$
- Maximum principle: $\max_{x\in V} L\mu(x)=\max_{x\in S(\mu)} L\mu(x)$

$\forall F\subset V$, $\nu^F=I(F)^{-1}\sigma^F$ where $(I(F),\sigma^F)$ is the solution of the linear programming problem

$$\min\ a\ ;\ \|\sigma\|=1;\ L\sigma\le a\mathbf{1}_F,\quad 0\le\sigma\le\mathbf{1}_F$$

L verifies the equilibrium principle

$\forall F\subset V$ there exists $\nu^F\in M(F)$ such that

$$L\nu^F(x)=1\ x\in F$$
$$L\nu^F(x)\le 1\ x\in V$$

Moreover $S(\nu^F)=F$

Finding ν^F

Second application

First Application

Finding Green functions

(See poster number 292)

Third application

SOLUTION IN TERMS OF EQUILIBRIUM MEASURES

$$u=\sum_{y\in A}\frac{\nu^{\{y\}\cup F}-\nu^F}{\nu^{\{y\}\cup F}(y)}$$

Condenser principle

$\forall A,B\subset V, A\neq\emptyset$ there exist $u\in C(V)$ such that

$$Lu(x)=0\quad 0\le u(x)\le 1\quad x\in F=(A\cup B)^c$$
$$Lu(x)\ge 0\quad u(x)=1\quad x\in A$$
$$Lu(x)\le 0\quad u(x)=0\quad x\in B$$

Dirichlet problem

$\forall A,B\subset V, A\neq\emptyset$ there exist $u\in C(V)$ such that

$$Lu(x)=0\quad x\in F$$
$$u(x)=1\quad x\in A$$
$$u(x)=0\quad x\in B$$

HITTING TIME

Definition:

The hitting time $H_y(x)$ is the expected number of steps for a reversible Markov chain before y is reached when started from state x.

State equation

$$\Delta H_y(x)=1\quad x\in V-\{y\}$$
$$H_y(y)=0$$

Dirichlet problem

$$LH_y(x)=c(x)\quad x\in V-\{y\}$$
$$H_y(y)=0$$

Poisson equation

$$LH_y(x)=c(x)-\delta_y(x)\sum_{z\in V}c(z)\quad x\in V$$

the unique solution such that $H_y(y)=0$

Explicit expression of $H_y(x)$

$$H_y(x)=\frac{1}{n}\sum_{z\in V}c(z)(\nu_y(x)+\nu_z(y)-\nu_z(x))$$

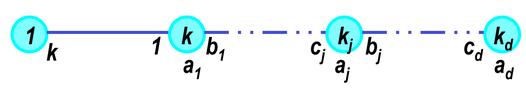
Regular graphs

$$H_y(x)=k\nu_y(x)$$

SOME EXAMPLES OF HITTING TIMES COMPUTED BY HAND

Distance-regular graph

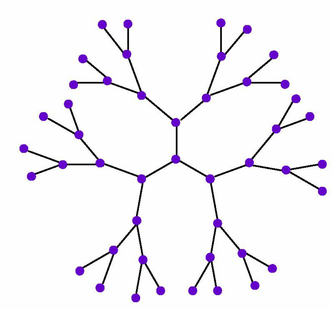
$$H_y(x)=k\sum_{j=0}^{d(x,y)-1}\frac{1}{|k_jb_j|}\sum_{l=j+1}^{d}k_l$$



Homogeneous tree of degree $k>2$ and deepness l

$$H_y(x)=\frac{1}{2}\sum_{z\in V}c(z)(d(x,y)+d(y,z)-d(z,x))$$

$$c(z)=\begin{cases}k & d(z,o)<l\\ 1 & d(z,o)=l\end{cases}$$



THE EFFECTIVE RESISTANCE

Definitions:

- The effective conductance between a and b, C_{ab}, is the value of the current from a to b if a and b are set at potential difference 1
- The effective resistance between a and b is $R_{ab}=C_{ab}^{-1}$
- The escape probability, P_{ab}, is the probability, starting at a, that a random walk reaches b before returning a, i.e., $P_{ab}=C_{ab}\,c(a)^{-1}$

State equation

$$\left.\begin{array}{ll}Lu(x)=0 & x\in V-\{a,b\}\\ u(a)=1 & \\ u(b)=0 & \end{array}\right\}\Rightarrow C_{ab}=I(u)$$

Explicit expression of R_{ab}

$$R_{ab}=\frac{1}{n}(\nu_a(b)+\nu_b(a))$$

EXTENSION TO NETWORK OF THE GENERAL EFFECTIVE RESISTANCE IN THE CONTINUOUS CASE

Let A, B, D and F disjoint subset of V, with $\delta(F)=A\cup B\cup D$. Then we define the effective conductance between A and B supposed D insulated as the energy of the solution of the following boundary value problem:

State equation

$$Lu(x)=0\quad x\in F$$
$$u(x)=1\quad x\in A$$
$$u(x)=0\quad x\in B$$
$$\frac{\partial u}{\partial\eta}(x)=0\quad x\in D$$

Equivalent state equation

$$\bar{L}u(x)=0\quad x\in F\cup D$$
$$u(x)=1\quad x\in A$$
$$u(x)=0\quad x\in B$$

where $\bar{L}$ is the Laplacian of $\bar{\Gamma}$ such that $\bar{L}u=Lu$ on F and $\bar{L}u=\frac{\partial u}{\partial\eta}$ on D

$$C^D_{AB}=I(u)=\langle Lu,u\rangle\quad\text{and}\quad u(x)=\sum_{y\in A}\frac{\nu^{\{y\}\cup F\cup D}-\nu^{F\cup D}}{\nu^{\{y\}\cup F\cup D}(y)}$$

SOME EXAMPLES OF EFFECTIVE RESISTANCE COMPUTED BY HAND

Distance-regular graph

$$R_{ab}=\frac{2}{n}\sum_{j=0}^{d(a,b)-1}\frac{1}{|k_jb_j|}\sum_{l=j+1}^{d}k_l$$

$$R_{a\delta(B_r(a))}=\sum_{j=0}^{r}\frac{1}{k_jb_j}$$

Infinite distance-regular graph with parameters k and l

$$p_{escape}=\lim_{r\to\infty}k^{-1}C_{a\partial(B_r(a))}=\frac{k-l-1}{k-l}$$

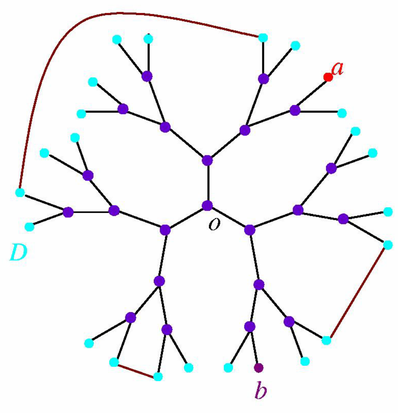
Homogeneous tree T_l^k of degree $k>2$ and deepness l

Let o the center of T_l^k, $H=\{x\in T_l^k: d(x,o)=l\}=a\cup b\cup D$.
Then the effective resistance between a and b supposed D insulated is given by

$$R^D_{ab}=d(a,b)$$

If we consider now a new graph in which any pair of vertices of H are connected, the solution of the state equation does not change, however the effective resistance of the new graph is given by

$$R^D_{ab}=\frac{4d^2(a,b)}{4d(a,b)+\sum_{\substack{x\sim y\\ x,y\in H}}(d(b,x)-d(a,x)+d(a,y)-d(b,y))^2}$$



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