Green's functions of self-adjoint operators on finite networks

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Settings and notations

- $\Gamma = (V, E, c)$ **network**
- **Combinatorial Laplacian** $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big) = \int_V c(x,y)\big(u(x) - u(y)\big)\, dy$$

- **Schrödinger operator** $\mathcal{L}_q : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}_q(u)(x) = \mathcal{L}(u)(x) + q(x)u(x),$$

where q **is an arbitrary function in** $\mathcal{C}(V)$

- **Normalized Laplacian** $\Delta : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\Delta(u)(x) = \frac{1}{\sqrt{\bar{c}(x)}} \int_V \bar{c}(x,y) \left(\frac{u(x)}{\sqrt{\bar{c}(x)}} - \frac{u(y)}{\sqrt{\bar{c}(y)}} \right) dy$$

If $c(x,y) = \dfrac{\bar{c}(x,y)}{\sqrt{\bar{c}(x)\bar{c}(y)}}$, **where** $\bar{c}(x) = \int_V \bar{c}(x,y)dy$ **and** $q_{\sqrt{\bar{c}}} = -\dfrac{\mathcal{L}(\sqrt{\bar{c}})}{\sqrt{\bar{c}(x)}}$

$$\Delta = \mathcal{L} + q_{\sqrt{\bar{c}}}$$

Boundary value problems

○ **Fixed** $q\in\mathcal{C}(V)$, $\emptyset\neq F\subset V$**,** $f\in\mathcal{C}(F)$**,** $g\in\mathcal{C}(\delta(F))$ **find** $u\in\mathcal{C}(\bar F)$

$$\left.\begin{aligned}\mathcal{L}_q(u)(x)&=f(x),\ \ \textbf{if } x\in F,\\ u(x)&=g(x),\ \ \textbf{if } x\in\delta(F),\end{aligned}\right\}\textbf{[BVP]}$$

where $\delta(F)=\{x\in F^c:\exists\, y\in F,\, c(x,y)\neq 0\}$ **and** $\bar F=F\cup\delta(F)$

- $F\neq V$ **Dirichlet problem on** F
- $F=V$ **Poisson equation on** V

Let $\sigma\in\mathcal{C}^+(V)$ **such that** $\mathsf{supp}(\sigma)=V$ **and** $q_\sigma=-\dfrac{1}{\sigma}\,\mathcal{L}(\sigma)$

$$\mathcal{L}_q(u)(x)=\frac{1}{\sigma(x)}\int_V c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)}-\frac{u(y)}{\sigma(y)}\right)dy+\big(q(x)-q_\sigma(x)\big)u(x)$$

- $\mathcal{L}_q$ **is positive semidefinite iff** $q\geq q_\sigma$
- $\sigma=\sqrt{\bar c}$ **then** $\Delta=\mathcal{L}_q$ **with** $q=q_\sigma$

How much negative can be function q_σ?

- **σ is not constant, q_σ takes necessarily negative and positive values, since** $\int_V \sigma(x)\, q_\sigma(x)\, dx = -\int_V \mathcal{L}(\sigma)(x)\, dx = 0$**.**

$$c(x)\Big(\frac{\sigma_m}{\sigma_M} - 1\Big) \le q_\sigma(x) = \frac{1}{\sigma(x)} \sum_{y\in V} c(x,y)\big(\sigma(y) - \sigma(x)\big) \le c(x)\Big(\frac{\sigma_M}{\sigma_m} - 1\Big)$$

$$\boxed{q_\sigma(x) > -c(x)}$$

- $\bar{F}_i \subset F_{i+1}$ **for all** $i = 1, \dots, k-1$, $F_0 = \emptyset$ **and** $F_{k+1} = V$**. Let** $t \in (0,1)$**,** $\sigma(x) = 1$, **if** $x \in F_i, i$ **odd and** $\sigma(x) = t$, **if** $x \in F_i, i$ **even.**

$$q_\sigma(x) = a_i \sum_{y\in F_{i-1}\cup F_{i+1}} c(x,y), \;\; a_i = \begin{cases} (t-1), & \textbf{if } i \textbf{ odd}, \\ (\frac{1}{t} - 1), & \textbf{if } i \textbf{ even}. \end{cases}$$

- **If** $k = 1$, $q_\sigma < 0$ **in** $\delta(F^c)$**,** $q_\sigma > 0$ **in** $\delta(F)$ **and** $q_\sigma = 0$**, otherwise.**

For any proper subset $F \subset V$, there exists σ, $q_\sigma < 0$ in F

- $\emptyset \neq F \subset V$ **and** $\nu^F \in \mathcal{C}^+(F)$ **the equilibrium measure for** F**:**

$$\mathcal{L}\nu^F = 1 \text{ in } F \text{ and } \mathrm{supp}(\nu^F) = F$$

$C_F^{-1} = \max\limits_{x \in \delta(F^c)} \displaystyle\int_{\delta(F)} c(x,y)\,dy$, **we define** $\sigma = \nu^F + a\,\mathbf{1}_{F^c}$ **where** $0 < a < C_F$.
Then,

$$q_\sigma(x) = \begin{cases} -\dfrac{1}{\nu^F(x)}, & \textbf{if } x \in \overset{\circ}{F}, \\ -\dfrac{1}{\nu^F(x)}\Big(1 - a\displaystyle\int_{\delta(F)} c(x,y)dy\Big), & \textbf{if } x \in \delta(F^c). \end{cases}$$

Monotonicity of $\mathcal{L}_q$ and consequences

$\otimes$ **There exists** $\sigma \in \mathcal{C}^+(V)$ **with** $\mathsf{supp}(\sigma) = V$ **such that** $q \geq q_\sigma$

$\otimes$ $\emptyset \neq F$, **it is not simultaneously satisfied that** $F = V$ **and** $q = q_\sigma$

$\triangle$ **Monotonicity** **If** $\mathcal{L}_q(u) \geq 0$ **on** F **and** $u \geq 0$ **on** F^c, **then** $u \in \mathcal{C}^+(V)$

$\triangle$ **Problem [BVP] is equivalent to**

$$\left.\begin{array}{ll} \mathcal{L}_q(u)(x) = f(x) - \mathcal{L}_q(g)(x), & \textbf{if } x \in F, \\[2ex] u(x) = 0, & \textbf{if } x \in \delta(F). \end{array}\right\} [\textbf{BVP}]_0$$

$\triangle$ **Existence and uniqueness** **For each** $f \in \mathcal{C}(F)$ **there exist a unique** $u \in \mathcal{C}(F)$ **such that** $\mathcal{L}_q(u) = f$ **on** F. **Moreover, if** $f \in \mathcal{C}^+(F)$, **then** $u \in \mathcal{C}^+(F)$ **and** $\mathsf{supp}(f) \subset \mathsf{supp}(u)$

Monotonicity of $\mathcal{L}_q$ and consequences

△ **Condenser principle** F, $\delta(F)=\{A,B\}$ **and** $u\in\mathcal{C}(V)$ **! sol.**

$$\left.\begin{aligned} \mathcal{L}_q(u)(x)&=0, && \text{if } \ x\in F,\\ u(x)&=\sigma(x), && \text{if } \ x\in A,\\ u(x)&=0, && \text{if } \ x\in B. \end{aligned}\right\}$$

Then, $0\le u\le\sigma$ **on** V**,** $\mathcal{L}_q(u)\ge 0$ **on** A **and** $\mathcal{L}_q(u)\le 0$ **on** B**.**

△ **Minimum principle Suppose that** $q=q_\sigma$**. If** $\mathcal{L}_q(u)\ge 0$ **on** F**, then**

$$\min_{x\in\delta(F)}\left\{\frac{u(x)}{\sigma(x)}\right\}\le\min_{x\in F}\left\{\frac{u(x)}{\sigma(x)}\right\}$$

equality holds iff u **coincides with a multiple of** σ **on** $\bar{F}$**.**

△ **Maximum principle Suppose that** $q=q_\sigma$**. If** $\mathcal{L}_q(u)=0$ **on** F**, then**

$$\min_{z\in\delta(F)}\left\{\frac{u(z)}{\sigma(z)}\right\}\le\frac{u(x)}{\sigma(x)}\le\max_{z\in\delta(F)}\left\{\frac{u(z)}{\sigma(z)}\right\}$$

equality holds iff u **coincides with a multiple of** σ **on** $\bar{F}$**.**

The kernel associated with an Schrödinger operator

△ $\mathcal{L}_q(u)(x) = \int_V L_q(x,y)u(y)\,dy$

△ **Associated kernel**

$$L_q(x,y) = \begin{cases} -c(x,y), & \textbf{if } x \neq y \\ \dfrac{1}{\sigma(x)}\displaystyle\int_V c(x,z)\sigma(z)dz + q(x) - q_\sigma(x) & \textbf{if } x = y \end{cases}$$

- $\gamma \in \mathcal{M}(V) = \mathcal{C}(V)$ **its potential is** $L_q(\gamma)$ **and its energy**

$$\mathcal{I}(\gamma) = \frac{1}{2}\int_{V\times V} c(x,y)\sigma(x)\sigma(y)\left(\frac{\gamma(y)}{\sigma(y)} - \frac{\gamma(x)}{\sigma(x)}\right)^2 dxdy + \int_V \big(q(x) - q_\sigma(x)\big)\,\gamma^2(x)dx$$

△ L_q **verifies the energy and Frostman's maximum principles**

△ **Equilibrium principle** **There exist a unique positive measure** γ^F

$$\mathsf{supp}(\gamma^F) = F \textbf{ and } L_q(\gamma^F) = \mathbf{1} \textbf{ on } F$$

Green's Functions

$\triangle$ $G : \bar{F} \times F \longrightarrow \mathbb{R}$ **is the Green function for** F **if for each** $y \in F$

$$\left.\begin{aligned} \mathcal{L}_q(G_y^F) = \varepsilon_y \quad &\textbf{on} \quad F \\ G_y^F = 0 \quad &\textbf{on} \quad \delta(F) \end{aligned}\right\}$$

$$\boxed{G^F(x,y) = \frac{\gamma^F(y)}{||\gamma^F|| - ||\gamma^{F-\{y\}}||}\big(\gamma^F(x) - \gamma^{F-\{y\}}(x)\big)}$$

$\triangle$ **If** $q = q_\sigma$ **and** $F = V$**,** $G : V \times V \longrightarrow \mathbb{R}$ **is the Green function for** V

$$\mathcal{L}_q(G_y) = \varepsilon_y - \frac{1}{||\sigma^2||}\sigma(y)\sigma \ \ \textbf{on} \ \ V$$

- $q = 0 \Longrightarrow \mathcal{L}(G_y) = \varepsilon_y - \dfrac{\mathbf{1}}{n}$ **on** V
- γ^y **the equilibrium measure for** $F = V - \{y\}$

$$\boxed{G(x,y) = |V|^{-2}\Big(||\gamma^y|| - |V|\,\gamma^y(x)\Big), \quad x,y \in V}$$

Green's Functions for V when $q = q_\sigma$

△ **Let** $\alpha = ||\sigma||^{-1}||\sigma^2||^{-1}$ **and** $\beta = ||\sigma||^{-1}||\sigma^2||^{-2}$

$$\begin{aligned} G(x,y) = \; & \alpha\,\sigma(x)\,\sigma(y)\int_V \sigma(z)\,\gamma^y(z)\,dz - \sigma(y)\,\gamma^y(x)\,||\sigma||^{-1} \\ + \; & \alpha\,\sigma(y)\int_V \sigma^2(z)\,\gamma^z(x)\,dz - \beta\,\sigma(x)\,\sigma(y)\int_{V\times V}\sigma^2(z)\,\sigma(w)\,\gamma^z(w)\,dz\,dw \end{aligned}$$

Poisson kernel for a proper set F

△ $P : \bar{F}\times\delta(F) \longrightarrow \mathbb{R}$ **is the Poisson kernel for** F **if for each** $y \in \delta(F)$

$$\left.\begin{aligned} \mathcal{L}_q(P_y^F) = 0 \quad & \textbf{on} \;\; F \\ P_y^F = \varepsilon_y \quad & \textbf{on} \;\; \delta(F) \end{aligned}\right\}$$

$$\boxed{P^F(x,y) = \frac{\gamma^{F\cup\{y\}}(x) - \gamma^F(x)}{\gamma^{F\cup\{y\}}(y)}}$$

Green's Functions for V when $q = q_\sigma$

$\triangle$ **Effective resistance** $R(x,y) = \dfrac{\sigma(x)\,\gamma^x(y) + \sigma(y)\,\gamma^y(x)}{||\sigma||}, \qquad x, y \in V$

$\triangle$ **Poisson kernel**

$$\boxed{P^F(x,y) = \frac{\gamma^{F\cup\{y\}}(x) - \gamma^F(x)}{\gamma^{F\cup\{y\}}(y)}}$$

$\triangle$ **Let** $\alpha = ||\sigma||^{-1}||\sigma^2||^{-1}$ **and** $\beta = ||\sigma||^{-1}||\sigma^2||^{-2}$

$$\begin{aligned} G(x,y) = {} & \alpha\,\sigma(x)\,\sigma(y)\int_V \sigma(z)\,\gamma^y(z)\,dz - \sigma(y)\,\gamma^y(x)\,||\sigma||^{-1} \\ + {} & \alpha\,\sigma(y)\int_V \sigma^2(z)\,\gamma^z(x)\,dz - \beta\,\sigma(x)\,\sigma(y)\int_{V\times V} \sigma^2(z)\,\sigma(w)\,\gamma^z(w)\,dz\,dw \end{aligned}$$