Green's function and Poisson kernel on a Path

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Some motivation an references

✓ R.P. Agarwal, Difference equations and inequalities, 2000

✓ F. Chung, S.T. Yau, Discrete Green's functions,2000

✓ A. Jirari, Second-order Sturm-Liouville difference equations and orthogonal polynomials, 1995

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$\Longrightarrow$ Chip-firing

$$\mathcal{L}(f) = c_i - c_e \quad \text{on} \quad F$$

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$\Longrightarrow$ Chip-firing

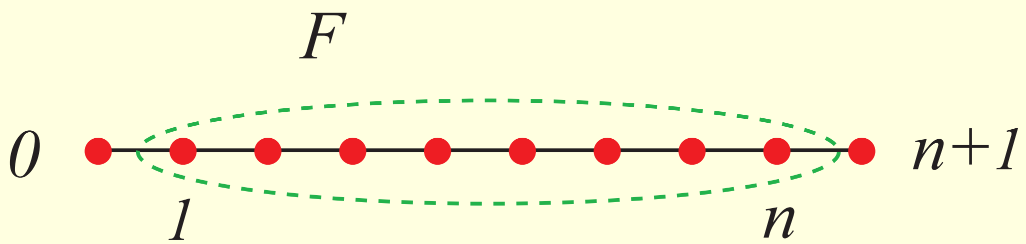
$$\mathcal{L}(f) = c_i - c_e \quad \text{on} \quad F$$

$\Longrightarrow$ Hitting-time

$$\mathcal{L}(H_k)(j) = \delta(j), \quad \text{on} \quad V \setminus \{k\}$$

Notations and definitions

- $\mathcal{P}_n$ a finite path on $n+1$ vertices



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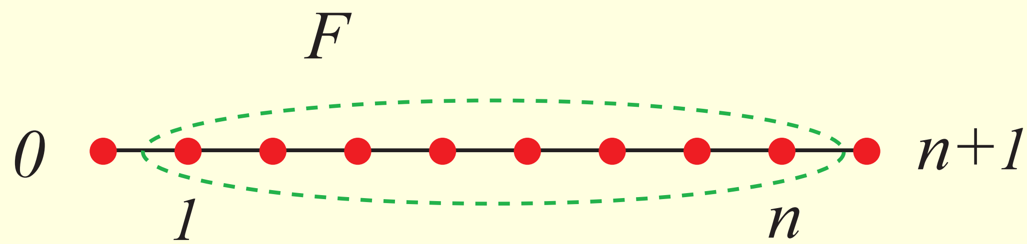

- The Schrödinger operator on $\mathcal{P}_n$

$$\mathcal{L}_q(u)(0) = (2q-1)u(0) - u(1)$$

$$\mathcal{L}_q(u)(k) = 2qu(k) - u(k+1) - u(k-1) \quad k \in F$$

$$\mathcal{L}_q(u)(n+1) = (2q-1)u(n+1) - u(n)$$

- Ground state: $2(q-1)$

Schödinger equation

- Schrödinger equation on F with data $f \in \mathcal{C}(F)$:

$$\boxed{\mathcal{L}_q(u) = f \quad \text{on} \quad F} \qquad \text{[ScE]}$$

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$$\boxed{\mathcal{L}_q(u) = 0 \ \text{ on } \ F} \qquad \text{[ScH]}$$

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- Wronskian of u and v:

$$w[u,v](k) = u(k)v(k+1) - u(k+1)v(k), \quad k = 0, \dots, n$$

$$w[u,v](n+1) = w[u,v](n)$$

Green function of [ScE]

✓ A Green's function is an integral kernel that can be used to solve an inhomogeneous differential equation.

✓ It serves roughly an analogous role in partial differential equations as does inverse matrix in the solution of linear systems.

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- $g_q \in \mathcal{C}(\bar{F} \times \bar{F})$

$$\mathcal{L}_q g_q(\cdot, s) = 0 \text{ on } F,\; g_q(s,s) = 0, g_q(s+1, s) = -1$$

$$\mathcal{L}_q g_q(\cdot, n+1) = 0 \text{ on } F,\; g_q(n+1, n+1) = 0, g_q(n, n+1) = 1$$

Green function of [ScE]

- ✓ If u and v are solution of [ScE], $w[u,v] =$ cte.
- ✓ u and v are l.i. iff $w[u,v] \neq 0$

$$g_q(k,s) = \frac{1}{w[u,v]}\Big[v(s)\,u(k) - u(s)\,v(k)\Big], \qquad k,s \in \bar{F}$$

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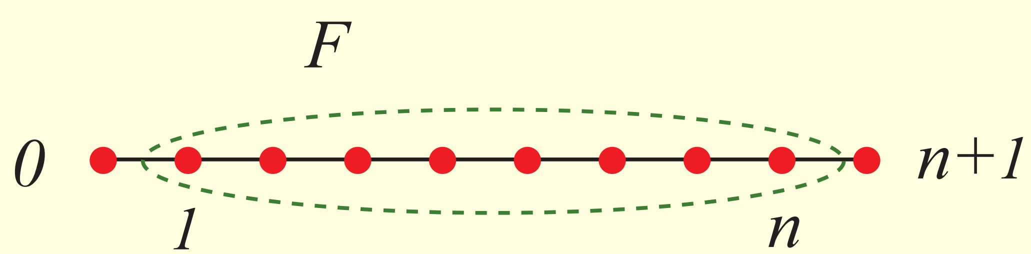

x'(0)=x(0)-x(1)

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$$x(k) = \frac{1}{w[u,v]} \Big[\Big(x_0 v(1) - x_1 v(0) \Big) u(k) - \Big(x_0 u(1) - x_1 u(0) \Big) v(k) \Big]$$

$$+ \sum_{s=1}^{k} g_q(k,s) f(s)\, ds$$

Green and Chebyshev

- $\{P_k\}_{k=-\infty}^{+\infty} \subset \mathbb{R}[x]$ is called Chebyshev sequence if

$$0 = 2\ x\ P_{k+1}(x) - P_{\ k}\ (x) - P_{k+2}(x)$$

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✓ $T_0(x) = 1, T_1(x) = x$ $\{T_k\}_{k=-\infty}^{+\infty}$ First kind

✓ $U_0(x) = 1, U_1(x) = 2x$ $\{U_k\}_{k=-\infty}^{+\infty}$ Second kind

✓ $V_0(x) = 1, V_1(x) = 2x - 1$ $\{V_k\}_{k=-\infty}^{+\infty}$ Third kind

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$$\Longrightarrow \quad \boxed{g_q(k,s) = -U_{k-s-1}(q), \qquad k,s \in \bar{F}}$$

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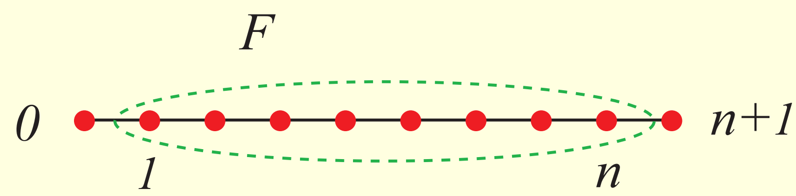
✓ The ! solution of [ScE] verifying $x(0) = x_0$, $x(1) = x_1$ is

$$\Longrightarrow x(k) = x_1 U_{k-1}(q) - x_0 U_{k-2}(q) - \sum_{s=1}^{k} U_{k-s-1}(q) f(s)\, ds$$

Sturm-Liouville Boundary conditions



- Sturm-Liouville Boundary conditions

$$\begin{bmatrix} \mathcal{U}_1(u) \\ \mathcal{U}_2(u) \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} u(n) \\ u(n+1) \end{bmatrix}$$

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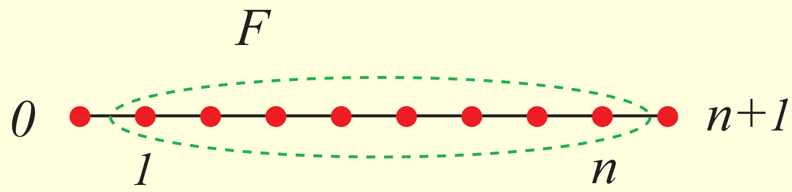

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- Sturm-Liouville BVP

$\mathcal{L}_q(u) = f$ on F $\quad au(0)+bu(1) = g_1, \;\; cu(n)+du(n+1) = g_2$

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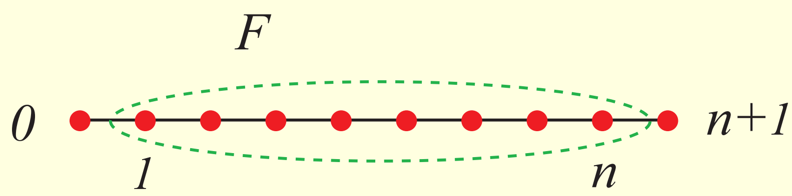

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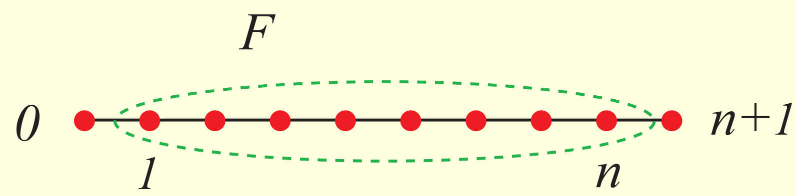
- Semi-Homogeneous Sturm-Liouville BVP

$\mathcal{L}_q(u) = f$ on $F \quad au(0) + bu(1) = cu(n) + du(n+1) = 0$

Sturm-Liouville Boundary conditions



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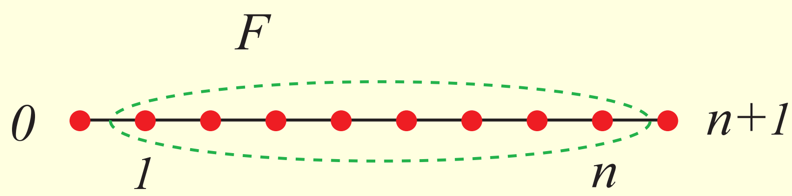
- Homogeneous Sturm-Liouville BVP

$\mathcal{L}_q(u) = 0$ on F $\quad au(0)+bu(1) = cu(n)+du(n+1) = 0 \qquad [H]$

Sturm-Liouville Boundary conditions



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- Sturm-Liouville BVP

$\mathcal{L}_q(u) = f$ on F $\quad au(0)+bu(1) = cu(n)+du(n+1) = 0 \ [SL]$

- The problem is regular iff [H] has as unique solution the trivial one

Green Function for [SL]

- Green function for [SL]: $G_q \in \mathcal{C}(\bar{F} \times F)$

$$\mathcal{L}_q(G_q(\cdot, s)) = \varepsilon_s \text{ on } F, \quad \mathcal{U}_1(G_q(\cdot, s)) = \mathcal{U}_2(G_q(\cdot, s)) = 0, s \in F$$

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- $W(q, A, B) = adU_n(q) + (ac + bd)U_{n-1}(q) + bcU_{n-2}(q)$

✓ The Sturm-Liouville problem is regular if $W(q, A, B) \neq 0$

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$\Longrightarrow$ If $0 \le k \le s \le n$

$$G_q(k,s) = \frac{\Big(aU_{k-1}(q) + bU_{k-2}(q)\Big)\Big(cU_{n-s-1}(q) + dU_{n-s}(q)\Big)}{adU_n(q) + (ac+bd)U_{n-1}(q) + bcU_{n-2}(q)}$$

$\Longrightarrow$ If $1 \le s \le k \le n+1$

$$G_q(k,s) = \frac{\Big(aU_{s-1}(q) + bU_{s-2}(q)\Big)\Big(cU_{n-k-1}(q) + dU_{n-k}(q)\Big)}{adU_n(q) + (ac+bd)U_{n-1}(q) + bcU_{n-2}(q)}$$

Dirichlet Problem

- Dirichlet Problem: $ab = cd = 0$

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$\Longrightarrow$ If $ad \neq 0$

The Dirichlet problem is regular if $q \neq \cos\left(\frac{k\pi}{n+1}\right)$

$\Longrightarrow$ In this case

$$G_q(k,s) = \frac{1}{U_n(q)} \begin{cases} U_{k-1}(q)\,U_{n-s}(q), & \text{si } \ 0 \leq k \leq s \leq n \\ U_{s-1}(q)\,U_{n-k}(q), & \text{si } \ 1 \leq s \leq k \leq n+1 \end{cases}$$

Neumann Problem

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$$\Longrightarrow \quad 0 \leq k \leq s \leq n$$

$$G_q(k,s)=\frac{V_{k-1}(q)\,V_{n-s}(q)}{2(q-1)\,U_{n-1}(q)}$$

$$\Longrightarrow \quad 1 \leq s \leq k \leq n+1$$

$$G_q(k,s)=\frac{V_{s-1}(q)\,V_{n-k}(q)}{2(q-1)\,U_{n-1}(q)}$$

Mixed Problem

- Dirichlet-Neumann Problem: $ab = 0$ and $c = -d \neq 0$

$$\Longrightarrow \quad \mathcal{U}_1(u) = au(0) \;\; \mathcal{U}_2(u) = d\Big(u(n+1) - u(n)\Big)$$

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The Mixed problem is regular if $q \neq \cos\left(\frac{(2k-1)\pi}{2n+1}\right)$

$\Longrightarrow \quad 0 \leq k \leq s \leq n$

$$G_q(k,s) = \frac{U_{k-1}(q)\, V_{n-s}(q)}{V_n(q)}$$

$\Longrightarrow \quad 1 \leq s \leq k \leq n+1$

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Other results

$\Longrightarrow$ One-point boundary condition

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$\Longrightarrow$ Poisson Kernel: $\mathcal{P}_q \in \mathcal{C}(\bar{F} \times \delta(F))$

$\mathcal{L}_q(\mathcal{P}_q(\cdot, 0)) = 0$ on F, $\mathcal{U}_1(\mathcal{P}_q(\cdot, 0)) = 1$, $\mathcal{U}_2(\mathcal{P}_q(\cdot, 0)) = 0$

$\mathcal{L}_q(\mathcal{P}_q(\cdot, n+1)) = 0$ on F, $\mathcal{U}_1(\mathcal{P}_q(\cdot, n+1)) = 0$, $\mathcal{U}_2(\mathcal{P}_q(\cdot, n+1)) =$ 1