On the $M$–property for distance–regular graphs

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**M-matrix**

Matrices with **non-positive** off-diagonal and **non-negative** diagonal entries

\[ L = kI - A \]

\( k > 0, A \geq 0 \) and the diagonal entries of A are less or equal to k
Matrices with non–positive off–diagonal and non–negative diagonal entries

\[ L = kI - A \]

\( k > 0, A \geq 0 \) and the diagonal entries of \( A \) are less or equal to \( k \)

If \( k \geq \rho(A) \), then \( L \) is called an \( M \)-matrix
Where can we find $M$-matrices?

- Finite difference methods for solving PDE
- Growth models in economics
- Markov processes in probability and statistics
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- Finite difference methods for solving PDE
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- Markov processes in probability and statistics
- ...the combinatorial Laplacian of a $k$-regular graph
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Symmetric and irreducible $M$–matrices

- **non–singular** $\rightarrow$ discrete Dirichlet problem $\rightarrow$ its inverse corresponds with the Green operator associated with the boundary value problem

- **singular** $\rightarrow$ discrete Poisson equation $\rightarrow$ its Moore–Penrose inverse corresponds with the Green operator too
Generalized inverses. The Moore–Penrose inverse

A **generalized inverse** of matrix $A$

- exists for a class of matrices larger than the class of non–singular matrices
- has some of the properties of the usual inverse
- reduces to the usual inverse when $A$ is non–singular

For every finite matrix $A$ there is a unique matrix $X$ satisfying the Penrose equations

$$AXA = A \quad (AX)^* = AX$$
$$XAX = X \quad (XA)^* = XA$$

$X$ is the **Moore–Penrose inverse**, and it is denoted by $A^\dagger$
Statement of the Problem

- If the $M$–matrix is non–singular its inverse has all entries positive
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- If the $M$–matrix is singular, it has a generalized inverse which is non–negative
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Given an $M$–matrix, there exists a generalized inverse which is also an $M$–matrix?
The Combinatorial Laplacian

When the Moore–Penrose inverse of the combinatorial Laplacian of a graph is an $M$-matrix?

$L^\dagger$ is an $M$-matrix iff $L^\dagger_{ij} \leq 0$ for any $x_i, x_j \in V$ with $i \neq j$
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Example: Petersen
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Network

- **Finite network**: $\Gamma = (V, E, c)$
- The **conductance**: $c: V \times V \rightarrow [0, +\infty)$ such that
  \[
  \begin{cases}
  c(x, x) = 0 & \text{for any } x \in V \\
  c(x, y) > 0 & \text{for } x \sim y.
  \end{cases}
  \]

- $C(V)$ is the set of real–valued functions on $V$
  - the functions in $C(V) \rightarrow \mathbb{R}^{|V|}$
  - the endomorphisms of $C(V) \rightarrow |V|$–order square matrices
The combinatorial Laplacian of $\Gamma$

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) \left( u(x) - u(y) \right), \quad x \in V$$

※ $\mathcal{L}$ is a positive semi–definite self–adjoint operator

※ $\mathcal{L}$ has 0 as its lowest eigenvalue whose associated eigenfunctions are constant

※ $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular $M$–matrix, $L$
Definition

Γ has the $M$–property iff $L^\dagger$ is a $M$–matrix.
Distance–regular graphs

Let $\Gamma$ be a distance–regular graph with intersection array

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, \ldots, c_D\}$$

$\Gamma$ is regular of degree $k$, and

$$k = b_0, \quad b_D = c_0 = 0, \quad c_1 = 1, \quad a_i + b_i + c_i = k$$
Examples

\begin{align*}
\text{n–Cycle: } \nu(C_n) &= \{2, 1, \ldots, 1; 1, \ldots, 1, c_D\} \\
\text{The Heawood Graph: } \nu(\Gamma) &= \{3, 2, 2; 1, 1, 3\} \\
\text{The Petersen Graph: } \nu(\Gamma) &= \{3, 2; 1, 1\}
\end{align*}
The intersection array $\iota(\Gamma)$

**Properties**

i) $k_0 = 1$ and $k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i}$, $i = 1, \ldots, D$

ii) $n = 1 + k + k_2 + \cdots + k_D$

iii) $k > b_1 \geq \cdots \geq b_{D-1} \geq 1$

iv) $1 \leq c_2 \leq \cdots \leq c_D \leq k$

v) If $i + j \leq D$, then $c_i \leq b_j$ and $k_i \leq k_j$ when, in addition, $i \leq j$

**Notation**

※ $a_1 = \lambda$

※ $c_2 = \mu$
Bipartite and Antipodal distance–regular graphs $\Gamma$

(i) $\Gamma$ is **bipartite** iff $a_i = 0$, $i = 1, \ldots, D$

(ii) $\Gamma$ is **antipodal** iff $b_i = c_{D-i}$, $i = 0, \ldots, D$, $i \neq \left\lfloor \frac{D}{2} \right\rfloor$

Distance–regular graphs with $k \geq 3$ other than bipartite and antipodal are **primitive**
The Moore–Penrose inverse of a distance–regular graph

Lemma

Let $\Gamma$ be a distance–regular graph.

\[
L^\dagger_{ij} = \sum_{r=d(x_i,x_j)}^{D-1} \frac{1}{nk_r b_r} \left( \sum_{l=r+1}^{D} k_l \right) - \sum_{r=0}^{D-1} \frac{1}{n^2 k_r b_r} \left( \sum_{l=0}^{r} k_l \right) \left( \sum_{l=r+1}^{D} k_l \right)
\]

for all $i, j = 1, \ldots, n$
The $M$–property

**Proposition**

A distance–regular graph $\Gamma$ has the $M$–property iff

$$\sum_{j=1}^{D-1} \frac{1}{k_j b_j} \left( \sum_{i=j+1}^{D} k_i \right)^2 \leq \frac{n-1}{k}$$

**Corollary**

*If $\Gamma$ has the $M$–property and $D \geq 2$, then*

$$\lambda \leq 3k - \frac{k^2}{n-1} - n$$

*and hence $n < 3k$.***
Small diameter

Proposition

If $\Gamma$ is a distance-regular graph with the $M$–property, then

$$D \leq 3$$

Proof.

If $D \geq 4$, then from property (v) of the parameters

$$3k < 1 + 3k \leq 1 + k + k_2 + k_3 \leq n$$
Diameter two

Proposition

A strongly regular graph with parameters \((n, k, \lambda, \mu)\) has the \(M\)-property iff

\[
\mu \geq k - \frac{k^2}{n-1}
\]

Observation

- Every antipodal strongly regular graph has the \(M\)-property
- Petersen graph, \((10, 3, 0, 1)\), does not have the \(M\)-property
Half of the strongly regular graphs have the \( M \)-property

Proposition

*If* \( \Gamma \) *is a primitive strongly regular graph*, then either \( \Gamma \) or \( \overline{\Gamma} \) *has the* \( M \)-*property.*

The graphs \( \Gamma \) and \( \overline{\Gamma} \), both of them, have the \( M \)-property iff \( \Gamma \) is a conference graph

**Conference Graph**

\((4m + 1, 2m, m - 1, m)\)
Diameter three

Proposition

A distance-regular graph with $D = 3$ has the $M$-property iff

$$k^2 b_1 \left( b_2 c_2 + (b_2 + c_3)^2 \right) \leq c_2^2 c_3^2 (n - 1)$$
Bipartite distance–regular graph with $D = 3$

\[ \nu(\Gamma) = \{k, k - 1, k - \mu; 1, \mu, k\} \]

**Proposition**

\( \Gamma \) satisfies the M–property iff

\[ \frac{4k}{5} \leq \mu \leq k - 1 \]

In particular, \( k \geq 5 \)

**Corollary**

When, \( 1 \leq \mu < k - 1 \), then either \( \Gamma \) or \( \Gamma_3 \) has the M–property, except when

\[ k - 1 < 5\mu < 4k \]

in which case none of them has the M–property
Antipodal distance–regular graph

\[ \nu(\Gamma) = \{k, t\mu, 1; 1, \mu, k\} \]

Proposition

An antipodal distance–regular graph with \( D = 3 \) has the \( M \)--property iff it is a Taylor graph \( T(k, \mu) \) such that \( k \geq 5 \) and

\[ \frac{k + 3}{2} \leq \mu < k \]
Corollary

If \( \Gamma \) is the Taylor graph \( T(k, \mu) \) with \( 1 \leq \mu \leq k - 2 \), then either \( \Gamma \) or \( \Gamma_2 \) has the \( M \)-property, except when

\[
\mu \in \{m - 2, m - 1, m, m + 1\} \quad \text{when} \quad k = 2m \quad \text{and}
\]

\[
\mu \in \{m - 1, m, m + 1\} \quad \text{when} \quad k = 2m + 1, \quad \text{in which case}
\]

none of them has the \( M \)-property.
Bipartite and antipodal distance–regular graph with $D = 3$

\[ \iota(\Gamma) = \{k, k - 1, 1; 1, k - 1, k\} \]

$k$–Crown graphs:

They are Taylor graphs with $\mu = k - 1$ and hence they have the $M$–property iff $k \geq 5$
Primitive case

Lemma

\[ k_2 = k \text{ iff } \Gamma \text{ is either } C_6, \ C_7 \text{ or } T(\mu, k) \]

Proposition

*If* \( \Gamma \) *satisfies the M–property, then*

\[ 1 < c_2 \leq b_1 < 2c_2 \quad b_2 < c_3 \quad k_3 \leq k - 3 \]

Moreover, \( k \geq 6 \) and \( c_2 < b_1 \) *when* \( \Gamma \) *is not a Taylor graph*

Corollary

*If* \( \Gamma \) *satisfies the M–property, then* \( b_2 < b_1 < k \) *and* \( 1 < c_2 < c_3 \)
Primitive case

Lemma

\[ k_2 = k \iff \Gamma \text{ is either } C_6, C_7 \text{ or } T(\mu, k) \]

Proposition

*If* \( \Gamma \) *satisfies the* M–*property, then*

\[
1 < c_2 \leq b_1 < 2c_2 \quad b_2 < c_3 \quad k_3 \leq k - 3
\]

*Moreover, \( k \geq 6 \) and \( c_2 < b_1 \) when \( \Gamma \) is not a Taylor graph*

Corollary

*If* \( \Gamma \) *satisfies the* M–*property, then* \( b_2 < b_1 < k \) *and* \( 1 < c_2 < c_3 \)
Facts

- None of the *Shilla graphs* satisfy the $M$–property
- None of the families of primitive distance–regular graphs with diameter 3 satisfy the $M$–property

Conjecture

*No primitive distance–regular graph with $D = 3$ satisfy the $M$–property.*
Introduction
The $M$–property
Distance–regular graphs

Properties
Characterization
Strongly regular graphs
Diameter three

Thank you!
The 5-Cycle has the $M$–property

The Moore–Penrose inverse of $L$ is an $M$–matrix

$$L = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix}$$
The 5-Cycle has the $M$–property

The Moore–Penrose inverse of $L$ is an $M$–matrix

$$L = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix}$$

$$L^\dagger = \begin{pmatrix}
2/5 & 0 & -1/5 & -1/5 & 0 \\
0 & 2/5 & 0 & -1/5 & -1/5 \\
-1/5 & 0 & 2/5 & 0 & -1/5 \\
-1/5 & -1/5 & 0 & 2/5 & 0 \\
0 & -1/5 & -1/5 & 0 & 2/5
\end{pmatrix}$$
The 5-Cycle has the $M$–property

The Moore–Penrose inverse of $L$ is an $M$–matrix

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$L^\dagger = \begin{pmatrix} 2/5 & 0 & -1/5 & -1/5 & 0 \\ 0 & 2/5 & 0 & -1/5 & -1/5 \\ -1/5 & 0 & 2/5 & 0 & -1/5 \\ -1/5 & -1/5 & 0 & 2/5 & 0 \\ 0 & -1/5 & -1/5 & 0 & 2/5 \end{pmatrix}$$
Petersen Graph does NOT have the $M$–property

The Moore–Penrose inverse of $L$ is NOT an $M$–matrix

\begin{align*}
(L^\dagger)_{ii} &= 0.33 \\
(L^\dagger)_{ij} &= 0.03 & \text{if } d(x_i, x_j) = 1 \\
(L^\dagger)_{ij} &= -0.07 & \text{if } d(x_i, x_j) = 2
\end{align*}