The $M$–property for distance–regular graphs

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**M-matrix**

Matrices with **non-positive** off–diagonal and **non-negative** diagonal entries

\[ L = kl - A \]

\( k > 0, A \geq 0 \) where diagonal entries of \( A \) are less or equal to \( k \).
M-matrix

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- Finite difference methods for solving PDE.
- Growth models in economics.
- Markov processes in probability and statistics.
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- ...the combinatorial Laplacian of a $k$-regular graph.
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> If \( k \geq \rho(A) \), then \( L \) is called an **\( M \)-matrix**
Where can find $M$-matrices?

Symmetric, irreducible $M$–matrices

- **non–singular** $\rightarrow$ discrete Dirichlet problem $\rightarrow$ its inverse corresponds with the Green operator associated with the boundary value problem.

- **singular** $\rightarrow$ discrete Poisson equation $\rightarrow$ its Moore–Penrose inverse corresponds with the Green operator too.
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- An irreducible and singular $M$–matrix has a generalized
Generalized inverses

$X$ is a **generalized inverse** of matrix $A$ if

- exists for a class of matrices larger than the class of singular matrices
- has some of the properties of the usual inverse
- reduces to the usual inverse when $A$ is nonsingular.
Moore–Penrose inverse

For every finite matrix $A$ there is a unique matrix $X$ satisfying the Penrose equations

\[
AXA = A, \quad (1)
\]
\[
XAX = X, \quad (2)
\]
\[
(AX)^* = AX, \quad (3)
\]
\[
(XA)^* =XA, \quad (4)
\]

where $A^*$ denotes the conjugate transpose of $A$. Matrix $X$ is commonly known as the Moore–Penrose inverse, and is denoted by $A^\dagger$. 
Preliminaries

Known results

- An irreducible and non-singular $M$-matrix has inverse with all entries positive.
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- An irreducible and non–singular $M$–matrix has inverse with all entries positive.

- An irreducible and singular $M$–matrix has a generalized inverse which is non–negative.

Explain Green, $L^\dagger$, $\Gamma^\dagger$
The equilibrium measure and the capacity

- The equilibrium measure There exists $\nu^x \in \mathcal{C}(V)$ such that
  \[
  \begin{cases}
  \nu^x(x) = 0 \\
  \nu^x(y) > 0 \quad y \neq x
  \end{cases}
  \]
  and
  \[\mathcal{L}(\nu^x) = 1 - n\varepsilon_x \quad \text{on } V.\]

  $\nu^x$ is the equilibrium measure of $V \setminus \{x\}$.

- The capacity is the function $\text{cap} \in \mathcal{C}(V)$ given by
  \[\text{cap}(x) = \sum_{y \in V} \nu^x(y).\]
Theorem

The Moore–Penrose inverse of $L$ is an $M$–matrix if, and only if, for any $x \in V$

$$\text{cap}(x) \leq n\nu^x(y) \quad \text{for any } y \sim x.$$
Proof

The Green function is given by

$$G(x, y) = \frac{1}{n^2} (\text{cap}(x) - n \nu^x(y)), $$

But, \( \min_{y \in V \setminus \{x\}} \{\nu^x(y)\} = \min_{y \sim x} \{\nu^x(y)\} \), since if the minimum is attained at \( z \not\sim x \),

$$1 = \mathcal{L}(\nu^x)(z) = \sum_{y \in V} c(x, y)(\nu^x(z) - \nu^x(y)) \leq 0.!! $$
Theorem

The network $\Gamma$ has the $M$–property iff for any $y \in V$

$$\text{cap}(y) \leq n\nu^{y}(x) \quad \text{for any } x \sim y.$$ 

In this case, $\Gamma$ is a subgraph of the subjacent graph of $\Gamma^\dagger$. 
Let $\Gamma$ be a distance–regular graph with intersection array

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, \ldots, c_D\}.$$

$\Gamma$ is regular of degree $k$, and

$$k = b_0, \quad b_D = c_0 = 0, \quad c_1 = 1, \quad a_i + b_i + c_i = k.$$
The intersection array $\iota(\Gamma)$

Properties

i) $k_0 = 1$ and $k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i}$, $i = 1, \ldots, D$.

ii) $n = 1 + k + k_2 + \cdots + k_D$.

iii) $k > b_1 \geq \cdots \geq b_{D-1} \geq 1$.

iv) $1 \leq c_2 \leq \cdots \leq c_D \leq k$.

v) If $i + j \leq D$, then $c_i \leq b_j$ and $k_i \leq k_j$ when, in addition, $i \leq j$.

Notation

$$a_1 = \lambda; \quad c_2 = \mu.$$
Examples

$n$–Cycle $\iota(C_n) = \{2, 1, \ldots, 1; 1, \ldots, 1, c_D\}$

The Heawood Graph
$\iota(\Gamma) = \{3, 2, 2; 1, 1, 3\}$

The Petersen Graph
$\iota(\Gamma) = \{3, 2; 1, 1\}$
Introduction

The $M$–property

Distance–regular graphs

Distance–regular graph $\Gamma$

(i) $\Gamma$ is bipartite iff $a_i = 0$, $i = 1, \ldots, D$.

(ii) $\Gamma$ is antipodal iff $b_i = c_{D-i}$, $i = 0, \ldots, D$, $i \neq \left\lfloor \frac{D}{2} \right\rfloor$

Distance–regular graphs with $k \geq 3$ other than bipartite and antipodal are primitive.
The equilibrium measure of a distance–regular graph

Lemma

Let $\Gamma$ be a distance–regular graph. Then, for all $x, y \in V$

$$\nu^x(y) = \frac{1}{\sum_{j=0}^{d(x,y)-1} \frac{1}{k_j b_j} \left( \sum_{i=j+1}^{D} k_i \right)}$$

$$\text{cap}(x) = \frac{1}{\sum_{j=0}^{D-1} \frac{1}{k_j b_j} \left( \sum_{i=j+1}^{D} k_i \right)^2}.$$
\( \Gamma \) has the \( M \)--property

Proposition

A distance--regular graph \( \Gamma \) has the \( M \)--property iff

\[
\sum_{j=1}^{D-1} \frac{1}{k_j b_j} \left( \sum_{i=j+1}^{D} k_i \right)^2 \leq \frac{n-1}{k}.
\]

Corollary

If \( \Gamma \) has the \( M \)--property and \( D \geq 2 \), then

\[
\lambda \leq 3k - \frac{k^2}{n-1} - n,
\]

and hence \( n < 3k \).
If the \( n \)–cycle \( C_n \) has the \( M \)–property \( \Rightarrow n < 6 \)

- \( D = 1 \Rightarrow n = 3 \)
- \( D = 2 \Rightarrow n = 4, 5 \).
If the \( n \)-cycle \( C_n \) has the \( M \)-property \( \Rightarrow n < 6 \)

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The Moore–Penrose inverse of \( C_n \) is

\[
(L^{\dagger})_{ij} = \frac{1}{12n} \left( n^2 - 1 - 6|i - j|(n - |i - j|) \right), \quad i, j = 1, \ldots, n,
\]
The Moore–Penrose inverse of $L$ is a $M$–matrix

\[
L^\dagger = (g_{ij}) = \begin{pmatrix}
2/5 & 0 & -1/5 & -1/5 & 0 \\
0 & 2/5 & 0 & -1/5 & -1/5 \\
-1/5 & 0 & 2/5 & 0 & -1/5 \\
-1/5 & -1/5 & 0 & 2/5 & 0 \\
0 & -1/5 & -1/5 & 0 & 2/5
\end{pmatrix}
\]
Small diameter

Proposition

If $\Gamma$ is a distance-regular graph with the $M$-property, then $D \leq 3$.

Proof.

If $D \geq 4$, then from property $\triangledown(v)$ of the parameters

$$3k < 1 + 3k \leq 1 + k + k_2 + k_3 \leq n,$$

\hfill \square
Proposition

A strongly regular graph with parameters \((n, k, \lambda, \mu)\) has the \(M\)-property iff

\[\mu \geq k - \frac{k^2}{n-1} .\]

Observation

- Every antipodal strongly regular graph has the \(M\)-property.
- The Petersen graph, \((10, 3, 0, 1)\), does not have the \(M\)-property.
Proposition

If $\Gamma$ is a primitive strongly regular graph, then either $\Gamma$ or $\bar{\Gamma}$ has the $M$–property.

The graphs $\Gamma$ and $\bar{\Gamma}$, both of them, have the $M$–property iff $\Gamma$ is a conference graph.
Partial geometries

A *partial geometry with parameters* \( s, t, \alpha \geq 1 \), \( pg(s, t, \alpha) \), *is an incidence structure of points and lines such that*

- every line has \( s + 1 \) points and every point is on \( t + 1 \) lines;
- two distinct lines meet in at most one point;
- given a line and a point not in it, there are exactly \( \alpha \) lines through the point which meet the line.

The number of points of \( pg(s, t, \alpha) \) is \( n = \frac{1}{\alpha}(s + 1)(st + \alpha) \)

The *point graph of* \( pg(s, t, \alpha) \), \( \Gamma \), *has the points as vertices and two vertices are adjacent iff they are collinear.*

\( \Gamma \) is a regular graph with degree \( k = s(t + 1) \).
Partial geometries and strongly regular graphs

Observation

- If $\alpha = s + 1$, the partial geometry is called *Linear space* and its point graph is the complete graph $K_n$.
- When $\alpha \leq s$, the point graph is a strongly regular graph with parameters $\left( n, s(t + 1), s - 1 + t(\alpha - 1), \alpha(t + 1) \right)$

A strongly regular graph $\Gamma$ is

- **geometric** if it is the point graph of a partial geometry;
- **pseudo–geometric** if its parameters are $\left( n, s(t + 1), s - 1 + t(\alpha - 1), \alpha(t + 1) \right)$, where $1 \leq \alpha \leq \min\{s, t + 1\}$ and $\alpha$ divides $st(s + 1)$

Not every pseudo–geometric graph is geometric.
Pseudo–geometric graphs and $M$–property

Corollary

A pseudo–geometric graph $\Gamma$ with parameters
\[ \left( n, s(t + 1), s - 1 + t(\alpha - 1), \alpha(t + 1) \right) \] has the $M$–property iff
\[ \alpha(2ts + t + \alpha) \geq st(s + 1). \]
Study of the point graphs associated to well–known families of partial geometries
Dual Linear Spaces

- \( \alpha = t + 1 \leq s \)
- \( \Gamma \) has the \( M \)-property iff \( s \leq 2(t + 1) \).

When, \( t = 1 \) and \( s = m - 2 \) the corresponding pseudo geometric graph is the *triangular graph* \( T_m \) with parameters \( \left( \binom{m}{2}, 2(m - 2), m - 2, 4 \right) \).

\( T_m \) has the \( M \)-property iff \( m = 4, 5, 6 \).
Transversal Designs and Dual Transversal Designs

- **Transversal Designs:**
  - \( \alpha = s \leq t + 1 \)
  - \( \Gamma \) is the complete multipartite graph \( K_{(s+1)\times(t+1)} \) whose parameters are
    \[
    \left( (s + 1)(t + 1), s(t + 1), (s - 1)(t + 1), s(t + 1) \right)
    \]
  - and it has the \( M \)-property.

- **Dual Transversal Designs:**
  - \( \alpha = t \leq s, \ t > 1 \)
  - \( \Gamma \) is the Pseudo–Latin square graph \( PL_{r}(m) \) whose parameters are
    \[
    \left( m^2, r(m - 1), r^2 - 3r + m, r(r - 1) \right)
    \]
    , where \( r = t + 1 \) and \( m = s + 1 \). It has the \( M \)-property iff \( s \leq 2t \).

For \( t = 2 \) it is the line graph of the complete bipartite graph \( K_{m,m} \), also called the squared lattice graph.
Generalized quadrangles

- $\alpha = 1, s > 1$
- $\Gamma$ is the pseudo-geometric graph

\[
((s + 1)(st + 1), s(t + 1), s - 1, t + 1).
\]

$\Gamma$ has the $M$–property iff $t(s^2 - s - 1) \leq 1$ and hence iff $t = 1$ and $s = 2$.

When $t = 1$: *Hamming graph* $H(2, s + 1)$ or *Lattice*.

Notice that the complement of $H(2, s + 1)$ is the pseudo-latin square graph $PL_s(s + 1)$ that satisfies the $M$–property.
Proper pseudo geometric geometric

(i) Kneser graphs $K(m, 2)$, where $m \geq 6$ is even, in which case $s = \frac{m}{2} - 1$, $t = m - 4$ and $\alpha = \frac{m}{2} - 2$. For arbitrary $m \geq 5$, the Kneser graph $K(m, 2)$ is the graph whose vertices represent the 2–subsets of $\{1, \ldots, m\}$, and where two vertices are connected if and only if they correspond to disjoint subsets. The parameters of the Kneser graph $K(m, 2)$ are $\left(\left(m\right)\binom{m-2}{2}, \left(m-4\right)\binom{m-3}{2}\right)$, that coincide with the parameters of the complement of $T_m$. Therefore, it has the $M$–property iff $m \geq 7$ as expected. In addition, $K(m, 2)$ for $m$ odd is an example of strongly regular graph that is not a pseudo geometric graph, which also implies that the complement of a pseudo geometric graph is not necessarily a pseudo geometric graph, see below.

(ii) ...
Questions

- When a strongly regular graph is pseudo geometric?
- When the complement of a pseudo-geometric graph is also pseudo-geometric?