Two-Side Boundary Value Problems in Distance-Regular Graphs

A. Carmona, A.M. Encinas and S. Gago

VIII Jornadas de Matemática Discreta

Almería, 11-13 Julio 2012
Outline

- Notations and definitions
  - Networks and matrices
  - Schrödinger equations
  - Boundary value problems on networks
  - Distance-regular graphs

- Boundary Value Problems on Paths
  - Homogeneous Schrödinger equation
  - Schrödinger equation with data \( f \in C(V) \)
  - Green’s matrix of a BVP
  - Two-side Boundary Value Problems
    - Unilateral boundary conditions
    - Sturm-Liouville boundary conditions
A network $\Gamma = (V, E, c)$ is composed by

- $V$ is a set of elements called vertices
- $E$ is a set of elements called edges
- $c : V \times V \rightarrow [0, \infty)$ is an application named conductance associated to the edges

Two vertices $i, j$ are adjacent, $i \sim j$ iff $c(i, j) \neq 0$

The degree of a vertex is $\omega_i = \sum_{j \in V} c(i, j)$
Matrices associated with networks

**Definition**

The **Laplacian** matrix of the network $\Gamma$ is defined as

$$(\mathcal{L})_{ij} = \begin{cases} 
\omega_i & \text{if } i = j, \\
-c(i, j) & \text{if } i \neq j, \\
0 & \text{otherwise.}
\end{cases}$$

**Definition**

A **Schrödinger matrix** $\mathcal{L}_Q$ on $\Gamma$ with potential $Q$ is defined as a generalization of the weighted Laplacian matrix as

$$\mathcal{L}_Q = \mathcal{L} + Q$$

where $Q = \text{diag}[q_0, \ldots, q_d]$ is the potential matrix.
### Definition

The **Laplacian** matrix of the network $\Gamma$ is defined as

$$
(L)_{ij} = \begin{cases} 
\omega_i & \text{if } i = j, \\
-c(i,j) & \text{if } i \neq j, \\
0 & \text{otherwise.}
\end{cases}
$$

### Definition

A **Schrödinger matrix** $L_Q$ on $\Gamma$ with potential $Q$ is defined as a generalization of the weighted Laplacian matrix as

$$
L_Q = L + Q
$$

where $Q = \text{diag}[q_0, \ldots, q_d]$ is the **potential matrix**.
Consider a subset of vertices $F \subset \Gamma(V)$.

The Schrödinger equation on $F$ with data $\vec{f}$ is the equation

$$(L_Q \vec{u}^T)_i = \vec{f}_i^T, \ i \in F, \ \vec{u}, \vec{f} \in \mathbb{R}^{n+1}$$

The homogeneous Schrödinger equation on $F$ is the equation

$$(L_Q \vec{u}^T)_i = 0, \ i \in F, \ \vec{u} \in \mathbb{R}^{n+1}$$
Green’s matrix of the Schrödinger equation

✓ Any solution $u$ of the homogeneous Schrödinger equation (HSE) satisfies the recurrence relation:

$$\lambda u_i(x) = a_i u_i(x) + b_{i-1} u_{i-1}(x) + c_{i+1} u_{i+1}(x), \quad 0 \leq i \leq d,$$

✓ The Wronskian of $u$ and $v \in \mathbb{R}^{n+1}$ is

$$w[u, v](k) = \begin{vmatrix} u_k & v_k \\ u_{k+1} & v_{k+1} \end{vmatrix}, \quad k = 0, \ldots, n,$$

$$w[u, v](n + 1) = w[u, v](n)$$

✓ Two solutions $u, v$ of the homogeneous Schrödinger equation are lin. independent iff $w[u, v](k) \neq 0$ for all $0 \leq k \leq n$. 

A. Carmona, A.M. Encinas and S.Gago

Two-Side BVP in Distance-Regular Graphs
The Green’s matrix of the Schrödinger equation, $G_Q$, is either the inverse matrix of $L_Q$ iff $L_Q$ is invertible, or the Moore-Penrose inverse of $L_Q$.

- For any $\vec{f} \in \mathbb{R}^{n+1}$ the unique solution of the Schrödinger equation with data $\vec{f}$ is

$$\vec{u} = G_Q \vec{f}.$$
Application of Schrödinger equations

We are interested in solving

- The Schrödinger equation on $F$ with data $f \in \mathbb{R}^{d+1}$:

  $$\mathcal{L}_Q u = f$$

- The Homogeneous Schrödinger equation on $F$:

  $$\mathcal{L}_Q u = 0$$

With certain linear conditions on the boundary

$$\mathcal{B}u = c_0 u_0 + c_1 u_1 + \cdots + c_d u_d = g \quad u \in \mathbb{R}^{d+1}.$$
Definition

A boundary value problem on $F$ consists in finding $u \in \mathbb{R}^{d+1}$ such that

$$\mathcal{L}Qu = f \text{ on } F, \quad B_1u = g_1, \quad B_2u = g_2,$$

for a given $f \in \mathbb{R}^{d+1}$ and $g_1, g_2 \in \mathbb{R}$.

Examples of application:

- $\mathcal{L}f = c_i - c_e$ on $F$ (Chip-firing games)

- $\mathcal{L}H_k = \delta_j$ on $V - \{k\}$ (Hitting-time)
Questions:

- Which kind of Schrödinger equations can we solve?
- Which kind of boundary value problems can we solve?
- In which kind of networks can we solve them?
Let $P_{n+2}$ be a finite path on $n + 2$ vertices, $V(P_{n+2}) = \overline{F}$.

**Definition**

The Schrödinger matrix on $P_{n+2}$ associated to \(\{p_i\}_{i=0}^{n+1}\) is

\[
L_Q(x) = \begin{pmatrix}
\frac{x-a_0}{k_0} & \frac{b_0}{k_1} & 0 & \ldots & \ldots & 0 \\
-\frac{b_0}{k_1} & \frac{x-a_1}{k_1} & \frac{-b_1}{k_2} & 0 & \ldots & 0 \\
0 & -\frac{b_1}{k_2} & \frac{x-a_2}{k_2} & -\frac{b_2}{k_3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & -\frac{b_n}{k_n} & \frac{x-a_{n+1}}{k_n}
\end{pmatrix}
\]

where $q_i(x) = \frac{x-a_i-b_{i-1}}{k_i} - \frac{b_i}{k_{i+1}}$ is the potential for any $0 \leq i \leq n + 1$. 

A. Carmona, A.M. Encinas and S.Gago

Two-Side BVP in Distance-Regular Graphs
Distance-regular graphs

- For two vertices \( u, v \in \Gamma \), the distance from \( u \) to \( v \), \( d(u, v) \), is the least number of edges in a path from \( u \) to \( v \).
- The maximum distance between two vertices of \( \Gamma \) is the diameter \( d \) of \( \Gamma \).
- Let

\[
(A_k)_{i,j} = \begin{cases} 
1 & \text{if } d(i,j) = k, \\
0 & \text{otherwise.}
\end{cases}
\]

be the distance matrices of \( \Gamma \), for \( 0 \leq k \leq d \). Observe that \( A_1 = A \) is the adjacency matrix of \( \Gamma \).
- A graph \( G = (V, E) \) is distance-regular iff there exists \( a_i, b_i, c_i \in \mathbb{R} \) such that

\[
AA_i = a_iA_i + b_{i-1}A_{i-1} + c_{i+1}A_{i+1}, \quad 0 \leq i \leq d,
\]

where \( b_{-1} = c_{d+1} = 0 \).
Distance-regular graphs

- The distance-matrices of a distance-regular are a polynomial of degree $i$ in $A$, that is, $A_i = p_i(A)$, $0 \leq i \leq d$. These polynomials are called the distance-polynomials, and they are a family of orthogonal polynomials satisfying the recurrence

\[ xp_i(x) = a_i p_i(x) + b_{i-1} p_{i-1}(x) + c_{i+1} p_{i+1}(x), \quad 0 \leq i \leq d, \]

- The associated matrix is a Jacobi matrix

\[
J_d = \begin{pmatrix}
a_0 & c_1 & 0 & \ldots & \ldots & 0 \\
b_0 & a_1 & c_2 & 0 & \ldots & 0 \\
0 & b_1 & a_2 & c_3 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & a_d
\end{pmatrix}
\]
Distance-regular graphs

Example: the cube

\[ J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \]
A boundary value problem on $F$ in $\Gamma$ is equivalent to a boundary value problem in a weighted path $P_{n+2}$.

$$J_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
3 & 0 & 2 & 0 \\
0 & 2 & 0 & 3 \\
0 & 0 & 1 & 0
\end{pmatrix}$$
Lemma 1

For any $k \in V$, the vectors $\vec{p} = (p_0(x), \ldots, p_d(x))$, $\vec{r} = (r_0(x), \ldots, r_d(x)) \in \mathbb{R}^{d+1}$ are a basis of the solution space of the HSE on $F$, as the wronskian is $w[\vec{p}, \vec{r}](n) = x/2$ for any $0 \leq n \leq d - 1$.

The Green matrix of the HSE is:

$$(G_H)_{ij} = \frac{2}{x} [p_i(x)r_j(x) - p_j(x)r_i(x)], \quad 0 \leq i, j \leq d, \quad x \in \mathbb{R}. \quad (1)$$

Thus, the general solution $\vec{y}$ of the Schrödinger equation on $F$ with data $\vec{f} \in \mathbb{R}^{d+1}$ is given for any $0 \leq i \leq d$ by

$$(\vec{y})_i = \alpha p_i(x) + \beta r_i(x) + \sum_{k=1}^{i} (G_H)_{ij} f_j, \quad \alpha, \beta \in \mathbb{R}.$$
A two-side boundary condition on a $P_{n+2}$ is

$$Bu = au_0 + bu_1 + cu_n + du_{n+1}, \text{ for any } u \in \mathbb{R}^{n+2}. $$

A two-side boundary value problem on $F$ consists in finding $u \in \mathbb{R}^{n+2}$ such that

$$\mathcal{L}_Q u = f \text{ on } F, \quad B_1 u = g_1, \quad B_2 u = g_2,$$

for a given $f \in \mathbb{R}^{n+2}$ and $g_1, g_2 \in \mathbb{R}$.

The problem is semi-homogeneous when $g_1 = g_2 = 0$, and homogeneous if besides $f = 0$ and $g_1 = g_2 = 0$. 
Two-side boundary value problems

- The boundary conditions in a matricial form:

\[
\begin{bmatrix}
B_1 \mathbf{u} \\
B_2 \mathbf{u}
\end{bmatrix}
= \begin{bmatrix}
c_{1,0} & c_{1,1} \\
c_{2,0} & c_{2,1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_0 \\
\mathbf{u}_1
\end{bmatrix}
+ \begin{bmatrix}
c_{1,n} & c_{1,n+1} \\
c_{2,n} & c_{2,n+1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_n \\
\mathbf{u}_{n+1}
\end{bmatrix}
\]

- Let $\mu_{ij}$ be the determinant of

\[
\mu_{ij} = \left| \begin{array}{cc}
c_{1i} & c_{1j} \\
c_{2i} & c_{2j}
\end{array} \right| , \quad j \in B = \{0, 1, n, n+1\}
\]

- The BVP is regular iff it has a unique solution iff the boundary polynomial

\[
P_c(x) = \sum_{i \in B} \sum_{j \in B} d_{ij} u_i v_j \neq 0,
\]
Two-side boundary value problems

By the classical theory of differential equations

✓ Any general BVP can be solved by solving the semihomogeneous BVP associated.

Definition

The Green’s matrix of the boundary value problem, $G_Q$, is the matrix such that

$$\mathcal{L}_Q G_Q = I, \quad \mathcal{B}_1 G_Q = 0, \quad \mathcal{B}_2 G_Q = 0$$

• For any $f \in \mathbb{R}^{n+2}$ the unique solution of the boundary value problem with data $f$ is

$$u = G_Q f.$$
The Green’s function of the two-side BVP

**Theorem**

The BVP is regular iff

\[ P_B(x) = \frac{x}{2} \sum_{i<j} \mu_{ij}(G_H(x))_{ij} \neq 0 \]

In this case, the Green matrix of the BVP problem, for any \( s \in F \), \( k \in V \), is the matrix whose \( ks \)-element, \((G_Q(x))_{ks}\), is given by

\[
\frac{x}{2P_B(x)} \left[ \frac{k_d-1}{c_d} \mu_{d-1}d(G_H(x))_{sk} + \sum_{i=0}^{1} (G_H(x))_{ik} \left( \sum_{j=d-1}^{d} \mu_{ij}(G_H(x))_{sj} \right) \right] \\
+ \left\{ \begin{array}{l}
0 \quad k \leq s, \\
(G_H(x))_{ks} \quad k \geq s.
\end{array} \right.
\]
Two-side boundary value problems

Typical two-side boundary value problems:

- **Unilateral BVP**
  - Initial value problem: $c_{2,j} = 0$ for $j \in B = \{0, 1, n, n + 1\}$
  - Final value problem $c_{1,i} = 0$ for $i \in B = \{0, 1, n, n + 1\}$

- **Sturm-Liouville BVP**

  \[ \mathcal{L}u = f \text{ on } F, \]
  \[ c_{1,0} u_0 + c_{1,1} u_1 = g_1, \]
  \[ c_{2,n} u_n + c_{2,n+1} u_{n+1} = g_2. \]

- outer-Dirichlet Problem $c_{1,0} c_{1,1} = c_{2,n} c_{2,n+1} = 0$.
- Neumann Problem $c_{1,0} + c_{1,1} = c_{2,n} + c_{2,n+1} = 0$.
- Dirichlet-Neumann Problem $c_{1,0} c_{1,1} = 0$, $c_{2,n} = -c_{2,n+1} \neq 0$. 
**Unilateral BVP**

Initial value problem: \( c_{2,j} = 0 \)  
Final value problem \( c_{1,i} = 0 \)

**Corollary 1**

The Green’s function for the initial value problem is

\[
(G_Q)_{k,s} = \begin{cases} 
0 & \text{if } k \leq s, \\
\frac{1+c_1}{Q_1(x)} \tilde{g}_x[k, s] & \text{if } k \geq s,
\end{cases}
\]

and the Green’s function for the final value problem is

\[
(G_Q)_{k,s} = \frac{1+c_1}{Q_1(x)} \tilde{g}_x[k, s] + \begin{cases} 
0 & \text{if } k \leq s, \\
\frac{1+c_1}{Q_1(x)} \tilde{g}_x[k, s] & \text{if } k \geq s,
\end{cases}
\]

where \( \tilde{g}_x[i, j] = [p_i(x)r_j(x) - p_j(x)r_i(x)] \), for \( i, j \in \overline{F} \).
Sturm-Liouville BVP

\[ au_0 + bu_1 = g_1, \quad cu_n + du_{n+1} = g_2 \text{ if } (|a| + |b|)(|c| + |d|) > 0 \]

**Corollary 2**

For the Sturm-Liouville conditions the boundary polynomial is

\[ P_c(x) = ac\tilde{g}_x[0, n] + ad\tilde{g}_x[0, n+1] + bc\tilde{g}_x[1, n] + bd\tilde{g}_x[1, n+1], \]

and the corresponding Green’s function for the Sturm-Liouville BVP is

\[
(G_Q)_{k,s} = \frac{1 + c_1}{Q_1(x)P_c(x)} \left( (a + xb)(c\tilde{g}_x[n, s] \right.
\left. + d\tilde{g}_x[n + 1, s])\tilde{g}_x[k, 0] \right) + \begin{cases} 
0 & \text{if } k \leq s, \\
g_x[k, s] & \text{if } k \geq s,
\end{cases}
\]

where \( \tilde{g}_x[i, j] = P_i(x)Q_j(x) - P_j(x)Q_i(x) \), for \( i, j \in \overline{F}. \)
Some References

Gracias

por su atención