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BOUNDS ON THE FIRST NONZERO EIGENVALUE FOR SELF-ADJOINT BOUNDARY VALUE PROBLEMS ON NETWORKS

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We aim here at obtaining bounds on the first nonzero eigenvalue for self-adjoint boundary value problems on a weighted network by means of equilibrium measures, that include the study of Dirichlet, Neumann and Mixed problems. We also show the sharpness of these bounds throughout the analysis of some examples. In particular we emphasize the case of distance-regular graphs and we show that the obtained bounds are better than those known until now.

1. INTRODUCTION

In this paper we analyze self-adjoint eigenvalue problems on a subset of a weighted network for the Laplace-Beltrami operator. Specifically, we concentrate on obtaining lower and upper bounds on the first nonzero eigenvalue associated with each problem.

Network eigenvalues have many applications in combinatorics and in other fields of mathematics. In the literature the mainly considered problems are those that concern with the Dirichlet eigenvalue problem and with the Poisson equation, see [**3**, **4**, **6**, **7**, **9**]. Some works involve the study of Neumann eigenvalues (see [**3**, **8**]), but no one considers the Dirichlet-Neumann boundary value problem. Here we firstly show that the study of eigenvalue problems can be reduced to the study of either a Dirichlet eigenvalue problem or a Poisson eigenvalue problem in a suitable network associated with the initial problem. Hence we obtain a new variational characterization of the first nonzero eigenvalue associated with each problem.

The techniques used here are the habitual in this context; that is to apply to a particular function a discrete version of the Green's Identity and the variational

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characterization of eigenvalues. The novelty lies in the functions that we consider, namely the equilibrium measure for suitable subsets of the network. The equilibrium measures in the context of finite networks were introduced by the authors in [**1**], where it was proved that these measures contain valuable information about the connection between vertices of a subset as well as the connection between the set and its complement. These properties were also made clear in [**2**] where we showed that the GREEN's Function of any self-adjoint boundary value problem on a network can be expressed in a simple form in terms of equilibrium measures. In addition, it must be noted that the equilibrium measure can be obtained either as the solution of a linear programming problem in which the Laplacian acts as the coefficient matrix of the general linear constraints or as the solution of a quadratic convex programming problem in which the Laplacian defines the objective function.

If one thinks about which functions are naturally associated with an arbitrary set in a general network, the DIRAC measures and the characteristic function of the set are more likely to be considered as candidates, but they only express whether a vertex is in or out of the set and they say nothing about the connectivity between vertices of the set. Therefore, if we try to consider functions that should take into account both aspects, the natural choice is not other one that the equilibrium measure of the set. We make the efficacy of this choice clear throughout some examples. Moreover we pay special attention to distance-regular graphs since in this type of graph the equilibrium measures can be computed by hand.

2. PRELIMINARIES

Throughout the paper $\Gamma = (V, E)$ denotes a simple and finite connected graph without loops, with vertex set V, edge set E, order n and size m. Two different vertices, $x, y \in V$, are called *adjacent*, which is represented by $x \sim y$, if $\{x, y\} \in E$. The cardinality of $F \subset V$ is denoted by $|F|$.

For each $x \in V$ and for each $j \in \mathbb{N}$ we denote by $S_j(x)$ and by $B_j(x)$ the *sphere* and the *ball of center x and radius j*; that is, the sets $S_j(x) = \{y \in V : d(x, y) = j\}$ and $B_j(x) = \{y \in V : d(x, y) \leq j\}$, where $d(x, y)$ is the length of the shortest path joining x and y.

Fix a vertex subset $F \subset V$. We denote by F^c its complement in V and we also consider the following vertex subsets associated with F:

(i) *Interior of F* : $\overset{\circ}{F} = \{x \in V : B_1(x) \subset F\}$.

(ii) *Boundary of F* : $\delta(F) = \{x \in V : d(x, F) = 1\}$.

(iii) *Closure of F* : $\bar{F} = \{x \in V : d(x, F) \leq 1\} = F \cup \delta(F)$.

A *dominating set* in Γ is a subset $F \subset V$ such that each element of F^c is adjacent to a vertex of F. Clearly a set is dominating iff $\bar{F} = V$ or equivalently $\overset{\circ}{F^c} = \emptyset$.

We denote by $\mathcal{C}(V)$ or $\mathcal{C}(V \times V)$ the sets of real functions defined on V and on $V \times V$, respectively. In addition for each nonempty set $F \subset V$ we denote by χ_F

its characteristic function and by $\mathcal{C}(F)$ the set of real functions on V that vanish in F^c. If $u \in \mathcal{C}(V)$, the value $\sum\limits_{x\in F} u(x)$ is denoted by $\int\limits_F u\,\mathrm{d}x$.

We call a *weighted network* a triple (Γ, c, ν), where $c \in \mathcal{C}(V\times V)$ is a symmetric function such that $c(x,y) > 0$ when $x \sim y$ and $c(x,y) = 0$ otherwise and $\nu \in \mathcal{C}(V)$ verifies that $\nu(x) > 0$ for each $x \in V$. If $x \in V$, the number $k(x) = \int\limits_V c(x,y)\,\mathrm{d}y$ is called the (*generalized*) *degree of* x. In addition, if $F \subset V$ is a proper subset, for any $x \in F$ the value $k_F^+(x) = \int\limits_{\delta(F)} c(x,y)\,\mathrm{d}y$ is called *out-degree of* x whereas, when $x \in F^c$, the value $k_F^-(x) = \int\limits_{\delta(F^c)} c(x,y)\,\mathrm{d}y$ is called *in-degree of* x. Observe that F is a dominating set iff $k_F^- > 0$ or, equivalently, iff $k_{F^c}^+ > 0$.

In what follows, if $F \subset V$ is nonempty, we consider for any $u \in \mathcal{C}(F)$ the values

$$\|u\|_{1,\nu} = \int\limits_F |u|\,\nu\,\mathrm{d}x \quad \text{and} \quad \|u\|_{2,\nu} = \Big(\int\limits_F u^2\,\nu\,\mathrm{d}x\Big)^{1/2}$$

and we define the *volume of* F as $\mathsf{vol}_\nu(F) = \|\chi_F\|_{1,\nu}$. We omit the subscript ν in all the above expressions when $\nu(x) = 1$ for all $x \in V$. In this case $\mathsf{vol}(F) = |F|$.

The *Laplace-Beltrami* operator of a weighted network (Γ, c, ν) is the linear operator $\mathcal{L} : \mathcal{C}(V) \to \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int\limits_V c(x,y)\big(u(x) - u(y)\big)\,\mathrm{d}y, \quad x \in V. \tag{1}$$

If $F \subset V$ is a nonempty subset, for each $u \in \mathcal{C}(\bar{F})$ we define the *conormal derivative of* u as the function belonging to $\mathcal{C}\big(\delta(F)\big)$ given by

$$\frac{\partial u}{\partial \mathsf{n}_F}(x) = \frac{1}{\nu(x)} \int\limits_{\delta(F^c)} c(x,y)\big(u(x) - u(y)\big)\,\mathrm{d}y, \quad x \in \delta(F). \tag{2}$$

In [**2**] the so-called *Green's Identity* was proved, namely

$$\int\limits_F v\mathcal{L}(u)\,\nu\,\mathrm{d}x - \int\limits_F u\mathcal{L}(v)\,\nu\,\mathrm{d}x = \int\limits_{\delta(F)} u\,\frac{\partial v}{\partial \mathsf{n}_F}\,\nu\,\mathrm{d}x - \int\limits_{\delta(F)} v\,\frac{\partial u}{\partial \mathsf{n}_F}\,\nu\,\mathrm{d}x, \quad u, v \in \mathcal{C}(\bar{F}). \tag{3}$$

For the case $\nu = 1$ the existence of the so-called equilibrium measures for any proper set was also proved. The techniques used there can easily be extended to the general case. Specifically for any proper subset $F \subset V$ there exists a unique function $\gamma^F \in \mathcal{C}(F)$, called *the equilibrium measure of* F, such that $\gamma^F(x) > 0$ for any $x \in F$ and $\mathcal{L}(\gamma^F) = 1$ on F. Moreover $\gamma^F = I(F)^{-1}\sigma^F$, where $(I(F)), \sigma^F)$ is the solution of the following quadratic convex programming problem:

$$I(F) = \min_{u\in\mathcal{C}(F)} \Big\{ \int\limits_F u\mathcal{L}(u)\,\nu\,\mathrm{d}x : u \geq 0,\ \|u\|_{1,\nu} = 1 \Big\}.$$

Alternatively the pair $(I(F)),\sigma^F)$ is also the solution of the linear programming problem

$$I(F)=\min_{u\in\mathcal{C}(F)}\{a: u\geq 0,\ \|u\|_{1,\nu}=1,\ \mathcal{L}(u)\leq a\chi_{F}\}.$$

The following result shows the relevance of the equilibrium measures in studying the topological properties of a subset.

Lemma 2.1. *If $F\subset V$ is a proper subset, then γ^F is constant on F iff k_F^+ is a positive multiple of ν.*

Proof. If $\gamma^F=a\chi_{F}$, then $1=\mathcal{L}(\gamma^F)=\dfrac{a}{\nu}\,k_F^+$ on F. Conversely, if $k_F^+=a\nu$, $a>0$, then $\gamma^F=\dfrac{1}{a}\,\chi_{F}$ is the equilibrium measure of F. □

Note that the above Lemma says that a necessary condition that the equilibrium measure of F be constant is that F^c is a dominating set.

Throughout the paper the so-called *distance-regular graph* plays an important role. Therefore we introduce here its definition and the value of some equilibrium measures associated with it, see [**2**].

A connected k-regular graph $\Gamma=(V,E)$ with diameter D is called distance-regular if there exist integers, b_i, c_i, $i=0,\dots,D$, such that for any two vertices $x,y\in V$ at distance $d(x,y)=i$ there are exactly c_i neighbors of x in $S_{i-1}(y)$ and b_i neighbors of x in $S_{i+1}(y)$. Then for any vertex $y\in V$ the values $|S_i(y)|$ and $|B_i(y)|$ do not depend on y and they are denoted by $k_i=|S_i|$ and $|B_i|$, respectively. Moreover $|B_i|=\sum\limits_{j=0}^{i}k_j,\ i=0,\dots,D$.

On the other hand the equilibrium measure of any ball in a distance-regular graph has the following expression

$$\gamma^{B_r}(x)=\sum_{s=|x|}^{r}\frac{|B_s|}{k_s b_s}\quad\text{for any}\quad x\in B_r, \tag{4}$$

where $|x|$ denotes the distance between x and the center of the ball. In addition, if γ_x denotes the equilibrium measure for the subset $V\setminus\{x\}$, for any $x,y\in V$ we have that

$$\gamma_x(y)=\sum_{j=0}^{d(x,y)-1}\frac{n-|B_j|}{k_j b_j}. \tag{5}$$

3. EIGENVALUES FOR SELF-ADJOINT BOUNDARY VALUE PROBLEMS

In [**2**] general self-adjoint boundary value problems were introduced in the context of finite networks and an exhaustive study of their associated Green functions was also carried out. In this paper we are concerned with another aspect of this type of problem, namely the study of eigenvalue problems.

Let (Γ, c, ν) be a weighted network and $F \subset V$ a nonempty connected subset with vertex boundary $\delta(F) = H_1 \cup H_2$, where $H_1 \cap H_2 = \emptyset$. A *self-adjoint eigenvalue problem on F for the Laplace-Beltrami operator, consists in finding $\lambda \in \mathbb{R}$ such that there exists $u \in \mathcal{C}(\bar{F})$ nonzero verifying*

$$\mathcal{L}(u) = \lambda\, u \text{ on } F, \quad \frac{\partial u}{\partial \mathsf{n}_F} = 0 \text{ on } H_1, \quad u = 0 \text{ on } H_2. \tag{6}$$

Problem (6) summarizes the different self-adjoint eigenvalue problems that appear in the literature with proper names; that is,

(i) *Dirichlet-Neumann eigenvalue problem* when $H_1, H_2 \neq \emptyset$.

(ii) *Dirichlet eigenvalue problem* when $H_2 = \delta(F) \neq \emptyset$.

(iii) *Neumann eigenvalue problem* when $H_1 = \delta(F) \neq \emptyset$.

(iv) *Poisson eigenvalue problem* when $\delta(F) = \emptyset$; that is when $F = V$.

It is well-known that the lowest eigenvalue of (6) is simple and nonnegative and that an eigenfunction can be chosen to be a positive function on $\mathcal{C}(F \cup H_1)$. Moreover the lowest eigenvalue is null for Neumann and Poisson problems and their corresponding eigenfunctions are constant on $\bar{F}$.

In the sequel we denote by $\lambda(F, H_1, H_2)$ the first nonzero eigenvalue for problem (6). It is also well-known that $\lambda(F, H_1, H_2)$ can be characterized from a variational point of view as

$$\lambda(F, H_1, H_2) = \min_{\substack{u \in \mathcal{C}(F \cup H_1) \\ u \neq 0}} \left\{ \frac{\int_F u\, \mathcal{L}(u)\, \nu\, \mathrm{d}x}{\int_F u^2 \nu\, \mathrm{d}x} : \frac{\partial u}{\partial \mathsf{n}_F} = 0 \text{ on } H_1 \text{ and } a \int_F u\nu\, \mathrm{d}x = 0 \right\}, \tag{7}$$

where $a = 1$ if $H_2 = \emptyset$, which corresponds to either Neumann or Poisson problems, and $a = 0$ otherwise.

The question of bounding the first nonzero eigenvalue for both the Dirichlet and Poisson problems on a network has been widely treated. However, this is not the case for the other eigenvalue problems, specially in the case of Dirichlet-Neumann problem the consideration of which is omitted in the literature. Some authors have dealt with the Neumann eigenvalue problem, see for instance [**3**, **5**, **8**], but the lower bounds for the eigenvalue are obtained only under strong constraints on the type of subsets considered.

Our objective is to determine bounds for $\lambda(F, H_1, H_2)$ in terms of equilibrium measures. For this we proceed analogously to [**2**] and hence we firstly reduce problem (6) to either a Dirichlet eigenvalue problem or a Poisson eigenvalue problem in a suitable network associated with the initial problem. The key idea is to notice that, if the conormal derivative of a function is null at $x \in \delta(F)$, then the value of the function at this vertex is uniquely determined by the values of the function on

F. Therefore in all cases the space of functions verifying the boundary conditions is isomorphic to $\mathcal{C}(F)$. Of course this result is only relevant when $H_1 \neq \emptyset$.

Lemma 3.1. *The function*

$$\boldsymbol{\eta}_{F,H_1} : \mathcal{C}(F) \to \mathcal{C}(F \cup H_1),$$

given by $\boldsymbol{\eta}_{F,H_1}(u) = u - \dfrac{\nu}{k_F^-} \dfrac{\partial u}{\partial \mathsf{n}_F} \chi_{H_1}$, *establishes an isomorphism between* $\mathcal{C}(F)$ *and* $\Big\{ u \in \mathcal{C}(F \cup H_1) : \dfrac{\partial u}{\partial \mathsf{n}_F} = 0$ *on* $H_1 \Big\}$.

From the above lemma and using the variational characterization of $\lambda(F, H_1, H_2)$ we obtain that

$$\lambda(F, H_1, H_2) = \min_{\substack{v \in \mathcal{C}(F) \\ v \neq 0}} \left\{ \frac{\int\limits_F v\, \mathcal{L}\big(\boldsymbol{\eta}_{F,H_1}(v)\big) \nu \, \mathrm{d}x}{\|v\|_{2,\nu}^2}, \quad a \int\limits_F v\nu \, \mathrm{d}x = 0 \right\} \tag{8}$$

since $\boldsymbol{\eta}_{F,H_1}(v) = v$ on F.

Next we show that the quadratic functional $\int\limits_F v\, \mathcal{L}(\boldsymbol{\eta}_{F,H_1}(v)) \nu \, \mathrm{d}x$ is in fact the quadratic functional associated with the Laplace-Beltrami operator of a suitable network with vertex set $F \cup H_2$. Therefore $\lambda(F, H_1, H_2)$ appears either as the first Dirichlet eigenvalue for the new network when $H_2 \neq \emptyset$ or as the first nonzero Poisson eigenvalue for the new network when $H_2 = \emptyset$. Specifically, given (Γ, c, ν) a weighted network and $F \subset V$ a proper connected subset, we define the function b: $(F \cup H_2) \times (F \cup H_2)$ as

$$b(x,y) = c(x,y) + \begin{cases} \displaystyle\int\limits_{H_1} \frac{c(x,z)c(y,z)}{k^-(z)} \, \mathrm{d}z, & x, y \in F \text{ and } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover we consider the weighted network $(\bar{\Gamma}_F, b, \nu)$ the vertex and edge sets of which are $F \cup H_2$ and $\bar{E} = \{\{x,y\} \in (F \cup H_2) \times (F \cup H_2) : b(x,y) > 0\}$, respectively. Note that in the new network the adjacency between vertices in $F \cup H_2$ is maintained, but a new adjacency can appear between vertices of $\delta(F^c)$ that have a common neighbor in H_1. Therefore F is a connected subset the boundary of which is now H_2. We also remark that, if (6) is a Poisson or a Dirichlet eigenvalue problem, then the network $(\bar{\Gamma}_F, b, \nu)$ coincides with the network (Γ, c, ν).

Proposition 3.2. *Let* (Γ, c, ν) *be a weighted network,* $\mathcal{L}$ *its Laplace-Beltrami operator and* $F \subset V$ *a proper connected subset. Then for any* $u, v \in \mathcal{C}(F)$

$$\int\limits_F v\, \mathcal{L}\big(\boldsymbol{\eta}_{F,H_1}(u)\big) \nu \, \mathrm{d}x = \int\limits_F v\, \bar{\mathcal{L}}(u) \nu \, \mathrm{d}x,$$

where $\bar{\mathcal{L}}$ *is the Laplace-Beltrami operator of the weighted network* $(\bar{\Gamma}_F, b, \nu)$.

Proof. Clearly it suffices to prove the equality

$$\int_F \varepsilon_x \, \mathcal{L}\big(\boldsymbol{\eta}_{F,H_1}(\varepsilon_y)\big)\nu \,\mathrm{d}z = \int_F \varepsilon_x \, \bar{\mathcal{L}}(\varepsilon_y)\nu \,\mathrm{d}z$$

for any $x, y \in F$, where ε_x stands for the DIRAC measure on x. If we consider $x, y \in F$, then

$$\begin{aligned}\int_F \varepsilon_x \, \mathcal{L}\big(\boldsymbol{\eta}_{F,H_1}(\varepsilon_y)\big)\nu \,\mathrm{d}z &= \int_V c(x,z)\big(\varepsilon_y(x) - \boldsymbol{\eta}_{F,H_1}(\varepsilon_y)(z)\big)\,\mathrm{d}z \\ &= \left(k(x)\varepsilon_y(x) - \int_F c(x,z)\varepsilon_y(z)\,\mathrm{d}z - \int_{\delta(F)} c(x,z)\boldsymbol{\eta}_{F,H_1}(\varepsilon_y)(z)\,\mathrm{d}z\right) \\ &= \left(k(x)\varepsilon_y(x) - c(x,y) - \int_{H_1} \frac{c(x,z)c(y,z)}{k_F^-(z)}\,\mathrm{d}z\right).\end{aligned}$$

Therefore, if $x \neq y$, we obtain that

$$\int_F \varepsilon_x \, \mathcal{L}\big(\boldsymbol{\eta}_{F,H_1}(\varepsilon_y)\big)\nu \,\mathrm{d}z = -b(x,y) = \int_F \varepsilon_x \, \bar{\mathcal{L}}(\varepsilon_y)\nu \,\mathrm{d}z$$

whereas, when $x = y$,

$$\int_F \varepsilon_x \, \mathcal{L}\big(\boldsymbol{\eta}_{F,H_1}(\varepsilon_x)\big)\nu \,\mathrm{d}z = \left(k(x) - \int_{H_1} \frac{c(x,z)^2}{k_F^-(z)}\,\mathrm{d}z\right).$$

On the other hand

$$\begin{aligned}\int_F \varepsilon_x \, \bar{\mathcal{L}}(\varepsilon_x)\nu \,\mathrm{d}z &= \int_{F\cup H_2} b(x,y)\,\mathrm{d}y \\ &= \left(\int_{F\cup H_2} c(x,y)\,\mathrm{d}y + \int_{(F\setminus\{x\})\times H_1} \frac{c(x,z)c(y,z)}{k^-(z)}\,\mathrm{d}z\mathrm{d}y\right) \\ &= \left(\int_{F\cup H_2} c(x,y)\,\mathrm{d}y + \int_{H_1} \frac{c(x,z)}{k^-(z)}\Big(\int_F c(y,z)\,\mathrm{d}y\Big)\mathrm{d}z - \int_{H_1} \frac{c(x,z)^2}{k_F^-(z)}\,\mathrm{d}z\right) \\ &= \left(k(x) - \int_{H_1} \frac{c(x,z)^2}{k_F^-(z)}\,\mathrm{d}z\right). \qquad \square\end{aligned}$$

Applying now the above proposition to Identity (8) we obtain that

$$\lambda(F, H_1, H_2) = \min_{\substack{v\in\mathcal{C}(F)\\ v\neq 0}} \left\{ \frac{\int_F v\,\bar{\mathcal{L}}(v)\nu\,\mathrm{d}x}{\|v\|_{2,\nu}^2}, \quad a\int_F v\nu\,\mathrm{d}x = 0 \right\}; \tag{9}$$

that is, $\lambda(F, H_1, H_2)$ is the first DIRICHLET eigenvalue for F on the network $(\bar{\Gamma}_F, b, \nu)$ when $H_2 \neq \emptyset$ or it is the first nonzero POISSON eigenvalue of the network

$(\bar{\Gamma}_{F}, b, \nu)$ when $H_2 = \emptyset$. Indeed the equality between the bilinear forms considered in the above proposition implies that the self-adjoint boundary problems raised in any of the networks are equivalent. Specifically we get the following result.

Corollary 3.3. *Let (Γ, c, ν) be a weighted network, $F \subset V$ a proper connected subset and suppose that $\delta(F) = H_1 \cup H_2$, where $H_1 \cap H_2 = \emptyset$. Then $u \in \mathcal{C}(F)$ satisfies $\bar{\mathcal{L}}(u) = f$ on F iff $v = \boldsymbol{\eta}_{F,H_1}(u)$ satisfies $\mathcal{L}(v) = f$ on F, $\dfrac{\partial v}{\partial n_{F}} = 0$ on H_1. In addition λ is an eigenvalue and $u \in \mathcal{C}(F)$ is an associated eigenfunction on $(\bar{\Gamma}_F, b, \nu)$ iff λ is an eigenvalue and $\boldsymbol{\eta}_{F,H_1}(u)$ is an associated eigenfunction on (Γ, c, ν) for the eigenvalue problem* (6).

4. POISSON EIGENVALUES

In this section we give bounds for the first nonzero eigenvalue of the Laplace-Beltrami operator of a weighted network in terms of equilibrium measures. Recall that the Poisson eigenvalue problem can be formulated as finding $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(V)$ nonzero such that $\mathcal{L}(u) = \lambda\, u$ on V. Therefore Poisson eigenvalues are nothing else but the so-called *network's eigenvalues* and for this reason $\lambda(V, \emptyset, \emptyset)$ is usually denoted as $\lambda(\Gamma)$ and it contains valuable information about the connectivity of Γ and is often called the *algebraic connectivity of* Γ, [**4**].

The following result displays a generalization of the most popular lower and upper bounds for $\lambda(\Gamma)$. When $\nu = 1$, the upper bound becomes $\lambda(\Gamma) \leq \dfrac{n}{n-1} \min\limits_{x \in V}\{k(x)\}$ and was obtained by M. Friedler in [**4**] whereas, when $\nu = k$ and $c(x,y) = 1$ for $x \sim y$, the lower bound gives $\lambda(\Gamma) \geq \dfrac{1}{2Dm}$ and was obtained by F. Chung in [**3**].

Lemma 4. *If D is the diameter of Γ, then*

$$\frac{1}{D\,\mathsf{vol}_\nu(V)} \min_{x \sim y}\{c(x,y)\} \leq \lambda(\Gamma) \leq \min_{x \in V}\left\{\frac{k(x)}{\nu(x)}\, \frac{\mathsf{vol}_\nu(V)}{\mathsf{vol}_\nu(V) - \nu(x)}\right\}.$$

Proof. The upper bound follows by considering the function $u = \varepsilon_x - \dfrac{\nu(x)}{\mathsf{vol}_\nu(V)}$ in the variational characterization of $\lambda(\Gamma)$ for any $x \in V$. The lower bound follows the guidelines of [**3**, Lemma 1.9]. $\square$

The equilibrium measures that play an essential role in this section are the equilibrium measures for sets of the form $V \setminus \{x\}$ with $x \in V$. As for any $x \in V$ it is verified that $\mathcal{L}(\gamma_x) = 1$ on $V \setminus \{x\}$, applying the Green's Identity we obtain that

$$\mathsf{vol}_\nu(V) = \int_V \mathcal{L}(\gamma_x)\, \nu\, \mathrm{d}y + \nu(x)\big(1 - \mathcal{L}(\gamma_x)(x)\big) = \nu(x)\big(1 - \mathcal{L}(\gamma_x)(x)\big)$$

and therefore

$$\mathcal{L}(\gamma_x) = 1 - \frac{\mathsf{vol}_\nu(V)}{\nu(x)}\, \varepsilon_x.$$

Theorem 4.2. *If* (Γ, c, ν) *is a weighted network, then*

$$\min_{x\in V}\left\{\frac{\mathsf{vol}_\nu(V)}{\|\gamma_x\|_{1,\nu}}\right\} < \lambda(\Gamma) \le \min_{x\in V}\left\{\frac{\mathsf{vol}_\nu(V)\|\gamma_x\|_{1,\nu}}{\mathsf{vol}_\nu(V)\|\gamma_x\|_{2,\nu}^2 - \|\gamma_x\|_{1,\nu}^2}\right\}.$$

Proof. If we fix $x \in V$ and we take $v = \|\gamma_x\|_{1,\nu} - \mathsf{vol}_\nu(V)\,\gamma_x$, then $\int\limits_V v\nu\,\mathrm{d}y = 0$ and v is a nonzero function since $v(x) = \|\gamma_x\|_{1,\nu} > 0$. Moreover

$$\int\limits_V v\mathcal{L}(v)\,\nu\,\mathrm{d}y = \mathsf{vol}_\nu(V)^2\|\gamma_x\|_{1,\nu}, \qquad \|v\|_{2,\nu}^2 = \mathsf{vol}_\nu(V)^2\|\gamma_x\|_{2,\nu}^2 - \mathsf{vol}_\nu(V)\|\gamma_x\|_{1,\nu}^2$$

and hence the upper bound follows from (7).

On the other hand, if $u \in \mathcal{C}(V)$ is a nonzero eigenfunction, then applying Green's Identity we obtain that for each $x \in V$ it is verified that

$$\lambda(\Gamma)\int\limits_V u\,\gamma_x\,\nu\,\mathrm{d}y = \int\limits_V \mathcal{L}(u)\,\gamma_x\,\nu\,\mathrm{d}y = \int\limits_V u\,\left(1 - \frac{\mathsf{vol}_\nu(V)}{\nu(x)}\,\varepsilon_x\right)\nu\,\mathrm{d}y = -\mathsf{vol}_\nu(V)\,u(x).$$

Moreover, as $\lambda(\Gamma) > 0$, it is also true that

$$\mathsf{vol}_\nu(V)\,|u(x)| = \lambda(\Gamma)\left|\int\limits_V u\,\gamma_x\,\nu\,\mathrm{d}y\right| \le \lambda(\Gamma)\int\limits_V |u|\,\gamma_x\,\nu\,\mathrm{d}y, \quad x \in V.$$

Hence, when we take $x_0 \in V$ such that $|u(x_0)| = \max\limits_{x\in V}|u(x)|$, the result follows keeping in mind that $\int\limits_V |u|\,\gamma_{x_0}\,\nu\,\mathrm{d}y \le |u(x_0)|\,\|\gamma_{x_0}\|_{1,\nu}$. □

If the upper bound is attained, then there exists $x \in V$ such that $v = \|\gamma_x\|_{1,\nu} - \mathsf{vol}_\nu(V)\,\gamma_x$ is an eigenfunction and hence γ_x is constant. Moreover we have the following result.

Lemma 4.3. *Given* $x \in V$, γ_x *is constant iff* $v = \|\gamma_x\|_{1,\nu} - \mathsf{vol}_\nu(V)\,\gamma_x$ *is an eigenfunction. Moreover the value* $\dfrac{k(x)}{\nu(x)}\,\dfrac{\mathsf{vol}_\nu(V)}{\mathsf{vol}_\nu(V) - \nu(x)}$ *is the corresponding eigenvalue.*

Proof. Suppose that $\gamma_x = a\chi_{V\setminus\{x\}}$. Then from Lemma 2.1 $a\,c(x,y) = \nu(y)$ for any $y \ne x$. Moreover

$$1 - \frac{\mathsf{vol}_\nu(V)}{\nu(x)} = \mathcal{L}(\gamma_x)(x) = -\frac{k(x)a}{\nu(x)}.$$

Therefore $a = \dfrac{\mathsf{vol}_\nu(V) - \nu(x)}{k(x)}$. Let $v = \|\gamma_x\|_{1,\nu} - \mathsf{vol}_\nu(V)\,\gamma_x = a\big(\mathsf{vol}_\nu(V)\varepsilon_x - \nu(x)\big)$. Then for any $y \ne x$

$$\mathcal{L}(v)(y) = a\,\mathsf{vol}_\nu(V)\mathcal{L}(\varepsilon_x)(y) = -a\,\mathsf{vol}_\nu(V)\,\frac{c(x,y)}{\nu(y)} = \mathsf{vol}_\nu(V)\,\frac{1}{a\nu(x)}v(y)$$

whereas

$$\mathcal{L}(v)(x) = a\,\mathsf{vol}_\nu(V)\mathcal{L}(\varepsilon_x)(x) = a\,\mathsf{vol}_\nu(V)\,\frac{k(x)}{\nu(x)} = \mathsf{vol}_\nu(V)\,\frac{1}{a\nu(x)}v(x).$$

Conversely, if $v = \|\gamma_x\|_{1,\nu} - \mathsf{vol}_\nu(V)\,\gamma_x$ is an eigenfunction associated with $\lambda \neq 0$, then

$$\mathcal{L}(v) = -\frac{\mathsf{vol}_\nu(V)}{\nu(x)}\big(\nu(x) - \mathsf{vol}_\nu(V)\varepsilon_x\big) = \lambda\big(\|\gamma_x\|_{1,\nu} - \mathsf{vol}_\nu(V)\gamma_x\big)$$

and hence γ_x is constant. □

In the case of distance-regular graphs, Theorem 4.2 provides bounds on $\lambda(\Gamma)$ in terms of the parameters associated with the graph.

Proposition 4.4. *If* Γ *is a distance-regular graph, then*

$$\frac{n}{\sum\limits_{j=0}^{D-1}\dfrac{(n-|B_j|)^2}{k_jb_j}} < \lambda(\Gamma) \leq \frac{n\sum\limits_{j=0}^{D-1}\dfrac{(n-|B_j|)^2}{k_jb_j}}{\sum\limits_{j=0}^{D-1}\dfrac{|B_j|\,(n-|B_j|)^3}{k_j^2b_j^2} + 2\sum\limits_{0\leq i<j\leq D-1}\dfrac{|B_i|\,(n-|B_i|)(n-|B_j|)^2}{k_ik_jb_ib_j}}.$$

Proof. From (5) we get that $\|\gamma_x\|_1 = \sum\limits_{j=0}^{D-1}\dfrac{(n-|B_j|)^2}{k_jb_j}$ and

$$\|\gamma_x\|_2^2 = \sum_{j=0}^{D-1}\frac{(n-|B_j|)^3}{k_j^2b_j^2} + 2\sum_{0\leq i<j\leq D-1}\frac{(n-|B_i|)(n-|B_j|)^2}{k_ik_jb_ib_j}.$$

Therefore the result follows by applying Theorem 4.2. □

The above bounds are better than the well-known bounds for general graphs

$$\frac{1}{nD} \leq \lambda(\Gamma) \leq \frac{n}{n-1}k$$

since for a distance-regular graph $\dfrac{(n-1)^2}{k} \leq \|\gamma_x\|_1 \leq n^2D$.

4.1. Dirichlet eigenvalues

In this subsection we obtain bounds for the first Dirichlet eigenvalue of a proper subset $F \subset V$ in terms of the equilibrium measure of F. Recall that, if $F \subset V$ is a proper subset, from (6) the Dirichlet eigenvalue problem on F consists on finding $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(F)$ nonzero such that $\mathcal{L}(u) = \lambda\, u$ on F. For this reason $\lambda\big(F, \emptyset, \delta(F)\big)$ is usually denoted by $\lambda_d(F)$.

Next we obtain bounds on $\lambda_d(F)$ in terms of γ^F, the equilibrium measure of F. In spite of the simplicity of their proofs, compare for instance with the technique used by H. Urakawa in [**9**, Theorem 2.1], we see through some examples that they are tight bounds, which shows again the good properties of the equilibrium measures.

Theorem 4.5. *Let (Γ, c, ν) be a weighted network. Then for each proper subset $F \subset V$ the following inequalities hold*:

$$\min_{x\in F}\left\{\frac{1}{\gamma^F(x)}\right\} \le \lambda_{d}(F) \le \frac{\|\gamma^F\|_{1,\nu}}{\|\gamma^F\|_{2,\nu}^2}.$$

Moreover any of the above inequalities is an identity iff k_F^+ is a multiple of ν.

Proof. The upper bound follows directly by taking $u = \gamma^F$ in the variational characterization of $\lambda_{d}(F)$.

To obtain a lower bound for $\lambda_{d}(F)$, consider now u a positive eigenfunction corresponding to $\lambda_d(F)$. As $u \in \mathcal{C}(F)$ verifies that $\mathcal{L}(u) = \lambda_{d}(F)u$ on F, applying Green's Identity we obtain

$$\lambda_{d}(F)\int_F u\,\gamma^F \nu\,\mathrm{d}x = \int_F \gamma^F \mathcal{L}(u)\,\nu\,\mathrm{d}x = \int_F u\,\mathcal{L}\gamma^F \nu\,\mathrm{d}x = \|u\|_{1,\nu}$$

and hence

$$\frac{1}{\max\limits_{x\in F}\{\gamma^F(x)\}} \le \lambda_{d}(F) \le \frac{1}{\min\limits_{x\in F}\{\gamma^F(x)\}},$$

which in particular gives the claimed lower bound.

Finally the lower bound is attained iff γ^F is constant and the upper bound is attained iff γ^F is an eigenfunction and hence iff γ^F is constant. Therefore, the result follows from Lemma 2.1. □

Note that the upper bound in the above proposition is better than the elementary upper bound obtained in the end of its proof since

$$\frac{\|\gamma^F\|_{1,\nu}}{\|\gamma^F\|_{2,\nu}^2} \le \frac{\|\gamma^F\|_{1,\nu}}{\min\limits_{x\in F}\{\gamma^F(x)\}\,\|\gamma^F\|_{1,\nu}} = \max_{x\in F}\left\{\frac{1}{\gamma^F(x)}\right\}.$$

In fact both bounds coincide iff k_F^+ is a multiple of ν. In particular this happens for any F proper subset of a complete graph since $\lambda_d(F) = n - |F| = \dfrac{1}{\gamma^F}$.

The following simple example shows the sharpness of the above bounds. Let P_{n+2} be a path on $n+2$ vertices and $F = \{x_1, \dots, x_n\}$, where $k(x_i) = 2$ for any $i = 1, \dots, n$. Then $\gamma^F(x_i) = \dfrac{i(n+1) - i^2}{2}$, $i = 1, \dots, n$, and therefore

$$\|\gamma^F\|_1 = \frac{n(n+1)(n+2)}{12} \quad \text{and} \quad \|\gamma^F\|_2^2 = \frac{n(n+1)(n+2)((n+1)^2+1)}{120}.$$

If we apply the above theorem, we get

$$\frac{2}{\left\lceil \dfrac{(n+1)^2}{4} \right\rceil} \le \lambda_{d}(F) \le \frac{10}{(n+1)^2+1}$$

whereas it is well-known that $\lambda_{_d}(F) = 2 - 2\cos\Big(\dfrac{\pi}{n+1}\Big)$.

From (4) and the above Theorem we get bounds on the first Dirichlet eigenvalue of both the complementary of a vertex and a ball of a distance-regular graph.

Proposition 4.6. *Let Γ be a distance-regular graph, then for any $x \in V$,*

$$\frac{1}{\displaystyle\sum_{s=0}^{D-1} \frac{n-|B_s|}{k_s b_s}} < \lambda_{_d}(V \setminus \{x\}) < \frac{\displaystyle\sum_{j=0}^{D-1} \frac{(n-|B_j|)^2}{k_j b_j}}{\displaystyle\sum_{j=0}^{D-1} \frac{(n-|B_j|)^3}{k_j^2 b_j^2} + 2 \sum_{0 \le i < j \le D-1} \frac{(n-|B_i|)(n-|B_j|)^2}{k_i k_j b_i b_j}}.$$

Proposition 4.7. *Let Γ be a distance-regular graph. Then for each $1 \le r \le D-1$*

$$\frac{1}{\displaystyle\sum_{s=0}^{r} \frac{|B_s|}{k_s b_s}} < \lambda_{_d}(B_r) < \frac{\displaystyle\sum_{s=0}^{r} \frac{|B_s|^2}{k_s b_s}}{\displaystyle\sum_{i=0}^{r} k_i \left(\sum_{s=i}^{r} \frac{|B_s|}{k_s b_s}\right)^2}.$$

4.2. Neumann eigenvalues

In this subsection we study the Neumann eigenvalue problem on a proper subset $F \subset V$; that is, to find $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(\bar{F})$ nonzero such that $\mathcal{L}(u) = \lambda\, u$ on F and $\dfrac{\partial u}{\partial \mathsf{n}_F} = 0$ on $\delta(F)$. For this reason $\lambda\big(F, \delta(F), \emptyset\big)$ is usually denoted by $\lambda_{_N}(F)$. Moreover throughout this section we suppose that $|F| \ge 2$ since otherwise the problem becomes trivial.

From Corollary 3.3 and tacking into account the bounds given in Theorem 4.2 we obtain lower and upper bounds for $\lambda_{_N}(F)$ in terms of equilibrium measures of the associated weighted network $(\bar{\Gamma}_F, b, \nu)$.

Proposition 4.8. *Let (Γ, c, ν) be a weighted network and $F \subset V$ a proper subset. Then*

$$\min_{x\in F}\left\{\frac{\mathsf{vol}_\nu(F)}{\|\bar{\gamma}_x\|_{1,\nu}}\right\} \le \lambda_{_N}(F) \le \min_{x\in F}\left\{\frac{\mathsf{vol}_\nu(F)\|\bar{\gamma}_x\|_{1,\nu}}{\mathsf{vol}_\nu(F)\|\bar{\gamma}_x\|_{2,\nu}^2 - \|\bar{\gamma}_x\|_{1,\nu}^2}\right\},$$

where $\bar{\gamma}_x$ is the equilibrium measure for $F \setminus \{x\}$ in the weighted network $(\bar{\Gamma}_F, b, \nu)$.

Next we analyze the following nontrivial example. Consider T_k, the infinite k-homogeneous tree rooted at o, and $F = B_r(o)$. Then the new graph consist of the finite k-homogeneous tree rooted at o and depth r so that $|F| = \dfrac{k(k-1)^r - 2}{k-2}$.

In [**2**] it was proved that

$$\bar{\gamma}_y^F(x) = \frac{|F|}{2}\, d(x,y) + \left(\frac{|F|}{2} + \frac{1}{k-2}\right)(|y| - |x|) + \frac{1}{(k-2)^2}\Big((k-1)^{r+1-|y|} - (k-1)^{r+1-|x|}\Big)$$

which implies that

$$\|\bar{\gamma}_y^F\|_1 = |F|\Big(|F| + \frac{2}{k-2}\Big)|y| + \frac{2|F|}{(k-2)^2}(k-1)^{r+1-|y|} - \frac{|F|(k-1)^{r+1}}{(k-2)^2} - \frac{k}{(k-2)^3}\Big(r(k-1)^{r+1} - (r+1)(k-1)^r + 1\Big) - \frac{(k-1)^r}{(k-2)^2}\Big(k - 1 + rk\Big).$$

This function attains its maximum value at any vertex y such that $|y| = r$ and hence

$$\min_{y\in F}\left\{\frac{|F|}{\|\bar{\gamma}_y\|_1}\right\} = \frac{(k-2)^2(k(k-1)^r - 2)}{(k-1)^{2r}(kr(k-2) - (k-1)) + \mathcal{O}(k^{r+2})}.$$

On the other hand, if we take, $y = 0$, then we obtain

$$\frac{|F|\,\|\bar{\gamma}_o^F\|_1}{|F|\,\|\bar{\gamma}_o^F\|_2^2 - \|\bar{\gamma}_o^F\|_1^2} = \frac{k(k-2)^2(k-1)^{3r-1}\big(k(k-2)+1\big) + \mathcal{O}(k^{2r+4})}{(k-1)^{3r+1}\big(k(k-2)+2\big) + \mathcal{O}(k^{2r+3})}.$$

Definitely we get

$$\lambda_{N}(F) \in \mathcal{O}(k) \quad \text{and} \quad \lambda_{N}^{-1}(F) \in \mathcal{O}(rk^{r-1}).$$

4.3. Dirichlet-Neumann Eigenvalues

In this subsection we obtain bounds for the first Dirichlet-Neumann eigenvalue on a proper subset F with $\delta(F) = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$ and $H_1, H_2 \neq \emptyset$. Recall that the Dirichlet-Neumann problem can be formulated as finding $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(F \cup H_1)$ nonzero such that $\mathcal{L}(u) = \lambda\, u$ on F and $\dfrac{\partial u}{\partial \mathsf{n}_F} = 0$ on H_1.

From Corollary 3.3 and taking into account the bounds given in Theorem 4.5 we obtain lower and upper bounds for $\lambda(F, H_1, H_2)$ in terms of equilibrium measures of the associated weighted network $(\bar{\Gamma}_F, b, \nu)$.

Proposition 4.9. *Let (Γ, c, ν) be a weighted network, $F \subset V$ a proper subset and suppose that $\delta(F) = H_1 \cup H_2$, where $H_1 \cap H_2 = \emptyset$ and $H_1, H_2 \neq \emptyset$. Then*

$$\min_{x\in F}\left\{\frac{1}{\bar{\gamma}^F(x)}\right\} \le \lambda(F, H_1, H_2)) \le \frac{\|\bar{\gamma}^F\|_{1,\nu}}{\|\bar{\gamma}^F\|_{2,\nu}^2}$$

where $\bar{\gamma}^F$ is the equilibrium measure for F in the weighted network $(\bar{\Gamma}_F, b, \nu)$. Moreover any of the above inequalities is an identity iff $\overset{\circ}{F} = \emptyset$ and $\int_{H_2} c(x,y)\,\mathrm{d}y$ is a nonzero multiple of ν on F.

We finish this subsection by analyzing some examples that show the tightness of the bounds obtained.

(i) Let K_n be the complete graph on n vertices, $F \subset V$ a proper set and H_1, H_2 a nontrivial partition of $\delta(F)$. Then

$$b(x,y) = \begin{cases} \dfrac{|F| + |H_1|}{|F|}, & \text{if } x, y \in F \text{ and } x \neq y, \\ 1, & \text{otherwise} \end{cases}$$

and therefore $\bar{\gamma}^F = \dfrac{1}{|H_2|}\, \chi_{_F}$ which implies that $\lambda(F, H_1, H_2) = |H_2|$.

(ii) Let P_{n+2} be a path the vertices of which are labeled as $x_0, x_1, \dots, x_{n+1}$. Consider the set $F = \{x_1, \dots, x_n\}$ and the Dirichlet-Neumann problem on F with boundary conditions $u(x_0) = 0$ and $u(x_{n+1}) = u(x_n)$. Then the new network $\bar{P}_{n+2}$ is a path on $n+1$ vertices with Dirichlet condition $u(x_0) = 0$ and therefore $\bar{\gamma}^F(x_i) = \dfrac{i(2n+1) - i^2}{2}$, $i = 0, \dots, n$. Moreover

$$\|\bar{\gamma}^F\|_1 = \frac{1}{6}\, n(n+1)(2n+1) \quad \text{and} \quad \|\bar{\gamma}^F\|_2^2 = \frac{1}{30}\, n(n+1)(2n+1)(2n^2+2n+1)$$

which implies

$$\frac{2}{n(n+1)} \le \lambda(F, H_1, H_2) \le \frac{5}{2n^2+2n+1}.$$

On the other hand in this case $\lambda(F, H_1, H_2) = 2 - 2\cos\left(\dfrac{\pi}{2n+1}\right)$.

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