Boundary Value Problems on Networks: The Effective Resistance

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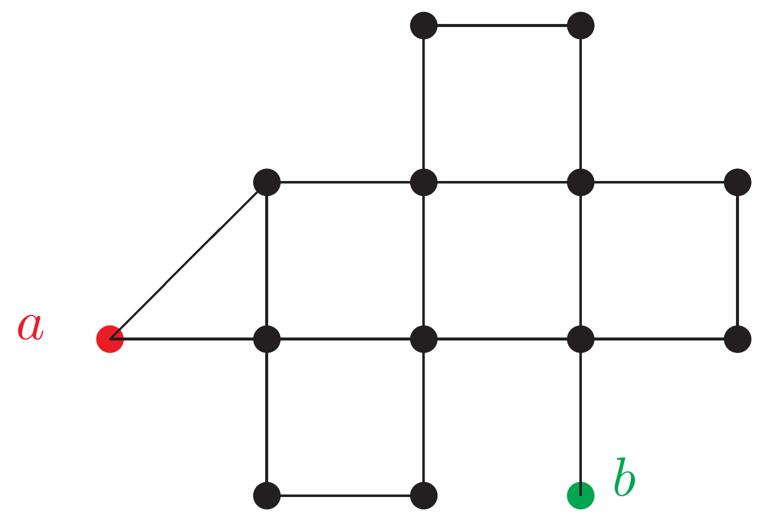
Networks: The Effective Resistance

○ $\Gamma = (V, E, c)$ **electrical network.**



$$\mathcal{L}u(x) = 0 \quad \textbf{if } x \in V - \{a, b\}$$

$$u(a) = 1 \qquad \Longrightarrow \qquad R_{ab} = I^{-1}(u) = \langle \mathcal{L}u, u \rangle$$

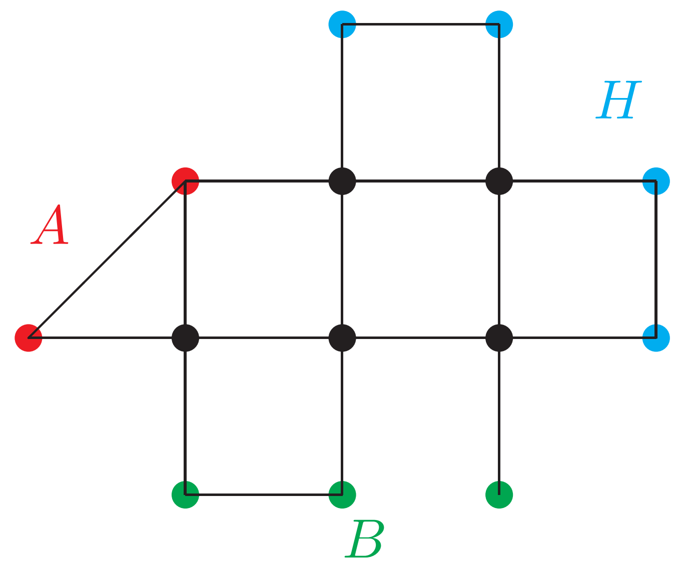
$$u(b) = 0$$

The Continuous Case

A plate conductor

$$\begin{cases} \mathsf{div}(\mathsf{c}\nabla(\mathsf{u})) = 0 & \\ \quad u = 1 & \textbf{on } A \\ \quad u = 0 & \textbf{on } B \\ \quad \frac{\partial u}{\partial \eta} = 0 & \textbf{on } C \textbf{ and } D \end{cases}$$

The Effective Resistance: A Generalization



$$\begin{cases} \mathcal{L}u = 0 & \textbf{on } F = V - (A \cup B \cup H) \\ u = 1 & \textbf{on } A \\ u = 0 & \textbf{on } B \\ \frac{\partial u}{\partial \eta} = 0 & \textbf{on } H \end{cases} \implies R_{AB} = I^{-1}(u)$$

$$\frac{\partial u}{\partial \eta}(x) = \sum_{\substack{y \sim x \\ y \in F}} c(x,y)\Big(u(x) - u(y)\Big)$$ normal derivative **of** u**.**

Boundary Value Problems on Networks

$$\left.\begin{aligned}
\mathcal{L}u(x) + q(x)u(x) &= f(x), \qquad x \in F \\
\frac{\partial u}{\partial \eta}(x) + h(x)u(x) &= g_1(x), \qquad x \in H_1 \\
u(x) &= g_2(x), \qquad x \in H_2
\end{aligned}\right\}$$

Potential Theory Techniques

- **Analitical Techniques**
  - **linear: Potentials**
  - **quadratic: Energy**
- **Symmentric Kernel**
  - **Green Kernel**

The Laplacian Kernel

- Equilibrium Problem

$$\exists!\ \nu \in \mathcal{M}(F) \text{ such that } \mathcal{L}\nu(x) = 1 \text{ on } F.$$

- Linear Programming Problem

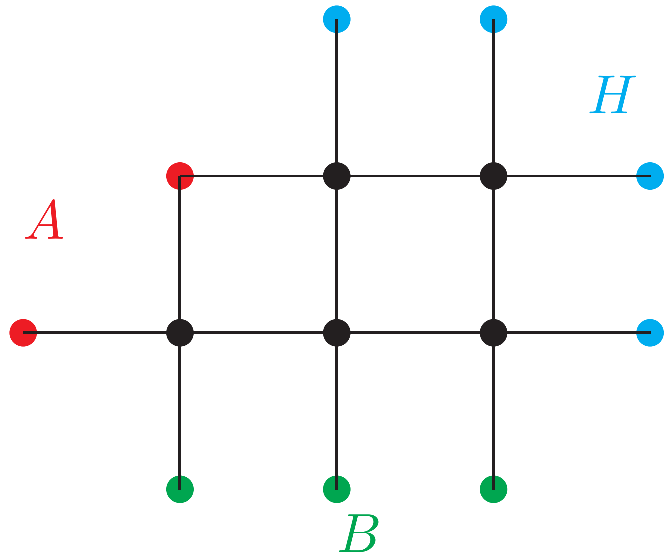
$$\min_{\mu \in \mathcal{M}^1(F)} \max_{x \in F} \mathcal{L}\mu(x) \iff \begin{array}{c} \min\{a\} \\ \mu(x) \geq 0 \\ \sum \mu(x) = 1 \\ \mathcal{L}_{|F}\mu(x) \leq a \end{array}$$

Solving a Boundary Value Problem

Let $\Gamma = (V, E, c)$ **and** $F \subset V$**. We define**

$$\bar{\Gamma}(F) = (\bar{F}, \bar{E}, \bar{c}),$$

where $\bar{F} = F \cup \delta(F)$ **and** $\bar{E} = \{(x, y) \in E : x \in F\}$



$$\bar{\mathcal{L}}u(x) = \mathcal{L}u(x) \quad \textbf{if } x \in F$$
$$\bar{\mathcal{L}}u(x) = \frac{\partial u}{\partial \eta}(x) \quad \textbf{if } x \in \delta(F)$$
$$\Longrightarrow$$
$$\bar{\mathcal{L}}u(x) + \bar{q}(x)u(x) = \bar{f}(x), \quad x \in F \cup H$$
$$u(x) = g_2(x), \quad x \in H_2$$

The Effective Resistance

$$\begin{cases} \mathcal{L}u = 0 & \textbf{on } F = V - (A \cup B \cup H) \\ u = 1 & \textbf{on } A \\ u = 0 & \textbf{on } B \\ \frac{\partial u}{\partial \eta} = 0 & \textbf{on } H \end{cases}$$

$$\Downarrow$$

$$\begin{aligned} \bar{\mathcal{L}}u(x) &= 0, \quad x \in F \cup H \\ u(x) &= 1, \quad x \in A \\ u(x) &= 0, \quad x \in B \end{aligned}$$

Condenser Principle

- **The** **condenser principle** **is satisfied if for any** $A, B \subset V$, $A \cap B = \emptyset$, $A \neq \emptyset$, **there exists** $u \in \mathcal{C}(V)$ **verifying:**

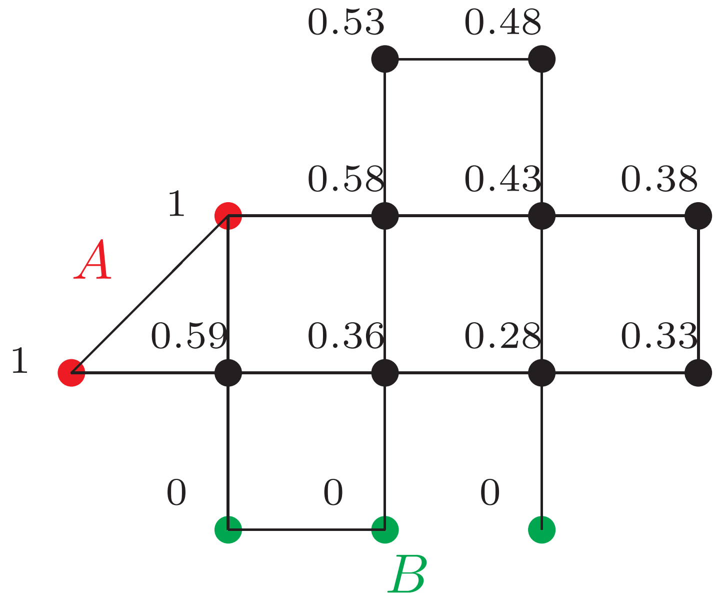
$$\begin{array}{lll} \mathcal{L}u(x) = 0, & 0 \le u(x) \le 1 & \textbf{if } x \in F = (A \cup B)^c, \\ \mathcal{L}u(x) \ge 0, & u(x) = 1 & \textbf{if } x \in A, \\ \mathcal{L}u(x) \le 0, & u(x) = 0 & \textbf{if } x \in B. \end{array}$$

- u **is solution iff** u **is solution of**

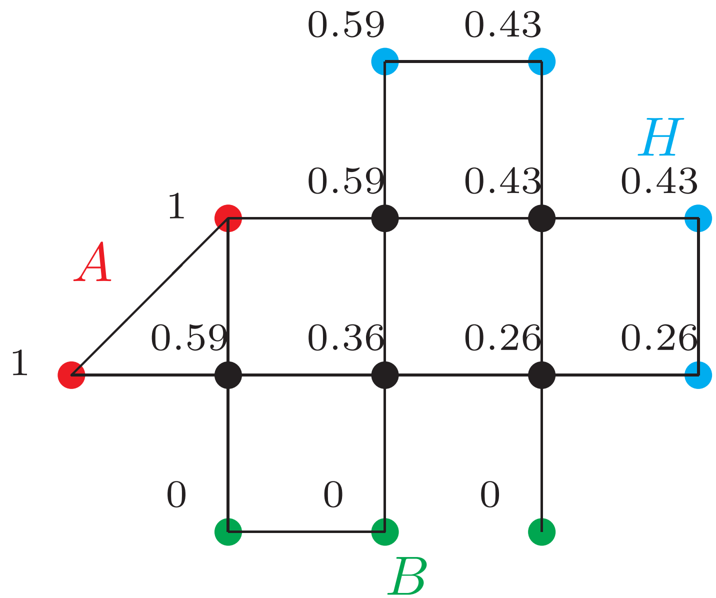
$$\begin{aligned} \mathcal{L}u(x) &= 0, & x &\in F = (A \cup B)^c \\ u(x) &= 1, & x &\in A \\ u(x) &= 0, & x &\in B \end{aligned}$$

$$u = \sum_{x \in A} \frac{\nu^{\{x\} \cup F} - \nu^F}{\nu^{\{x\} \cup F}(x)}$$

Our example

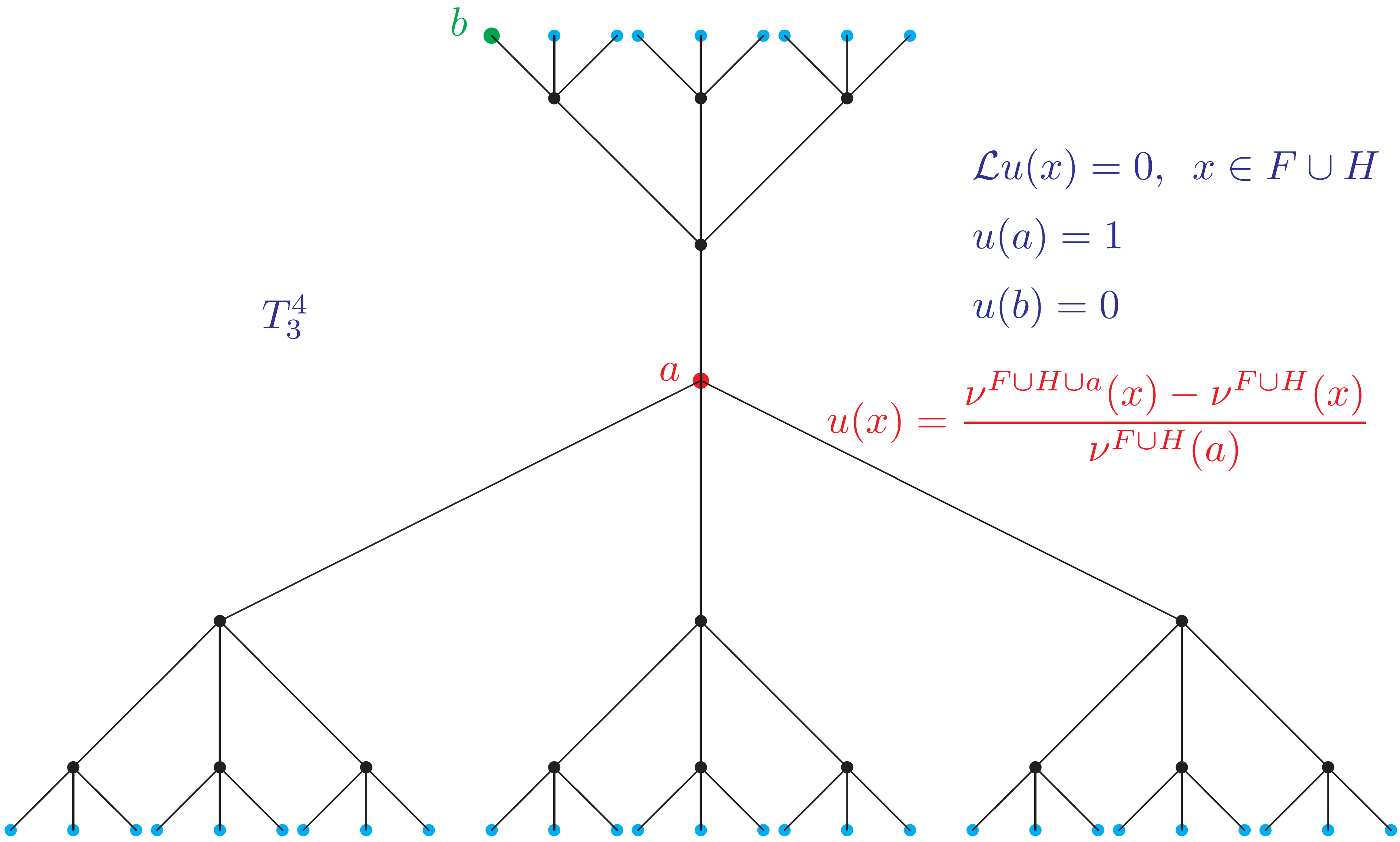


$R_{AB} = 0.81$



$R_{AB} = 0.78$

The Effective Resistance of a Tree: T_ℓ^k



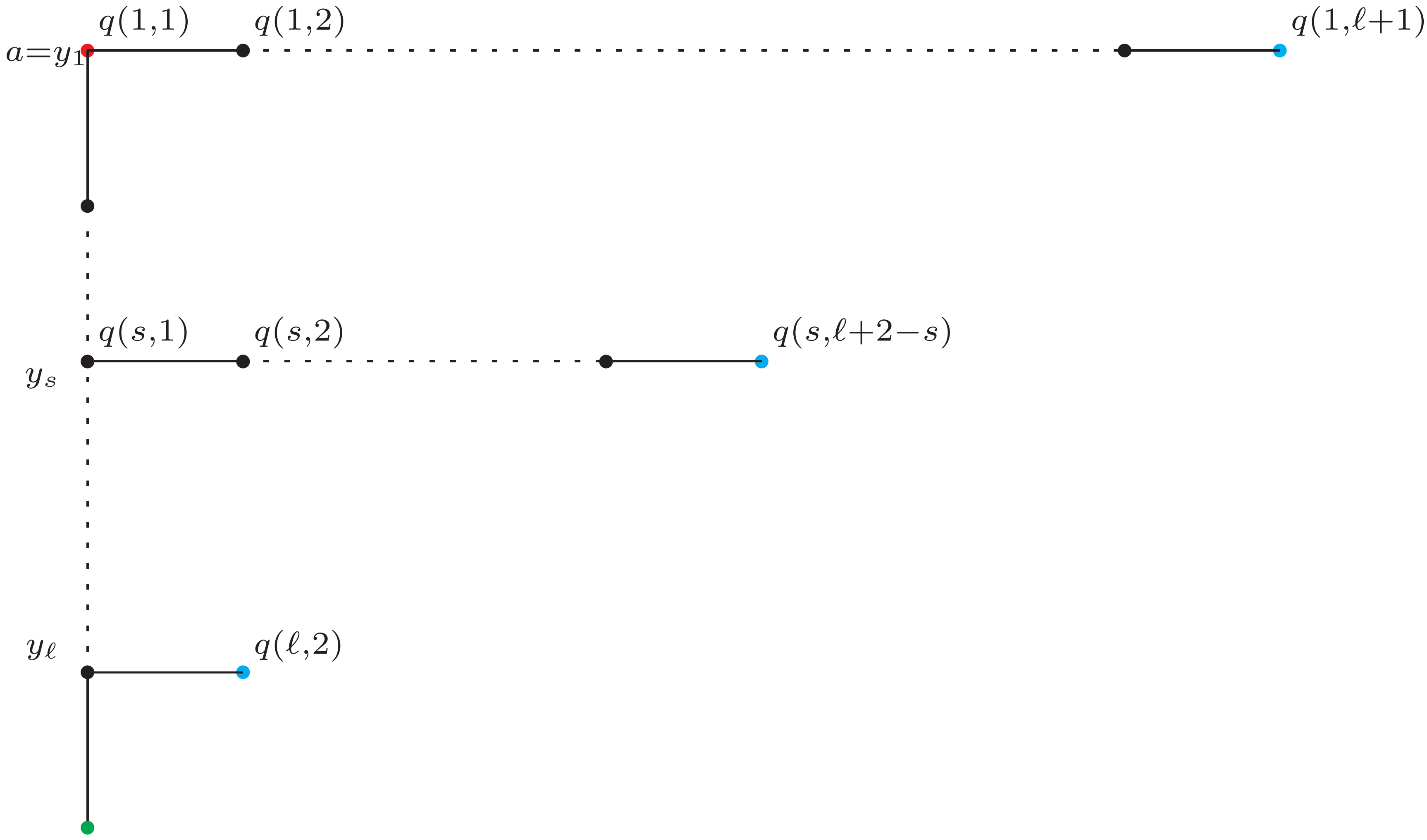
$$\mathcal{L}u(x) = 0, \quad x \in F \cup H$$

$$u(a) = 1$$

$$u(b) = 0$$

$$u(x) = \frac{\nu^{F\cup H\cup a}(x) - \nu^{F\cup H}(x)}{\nu^{F\cup H}(a)}$$

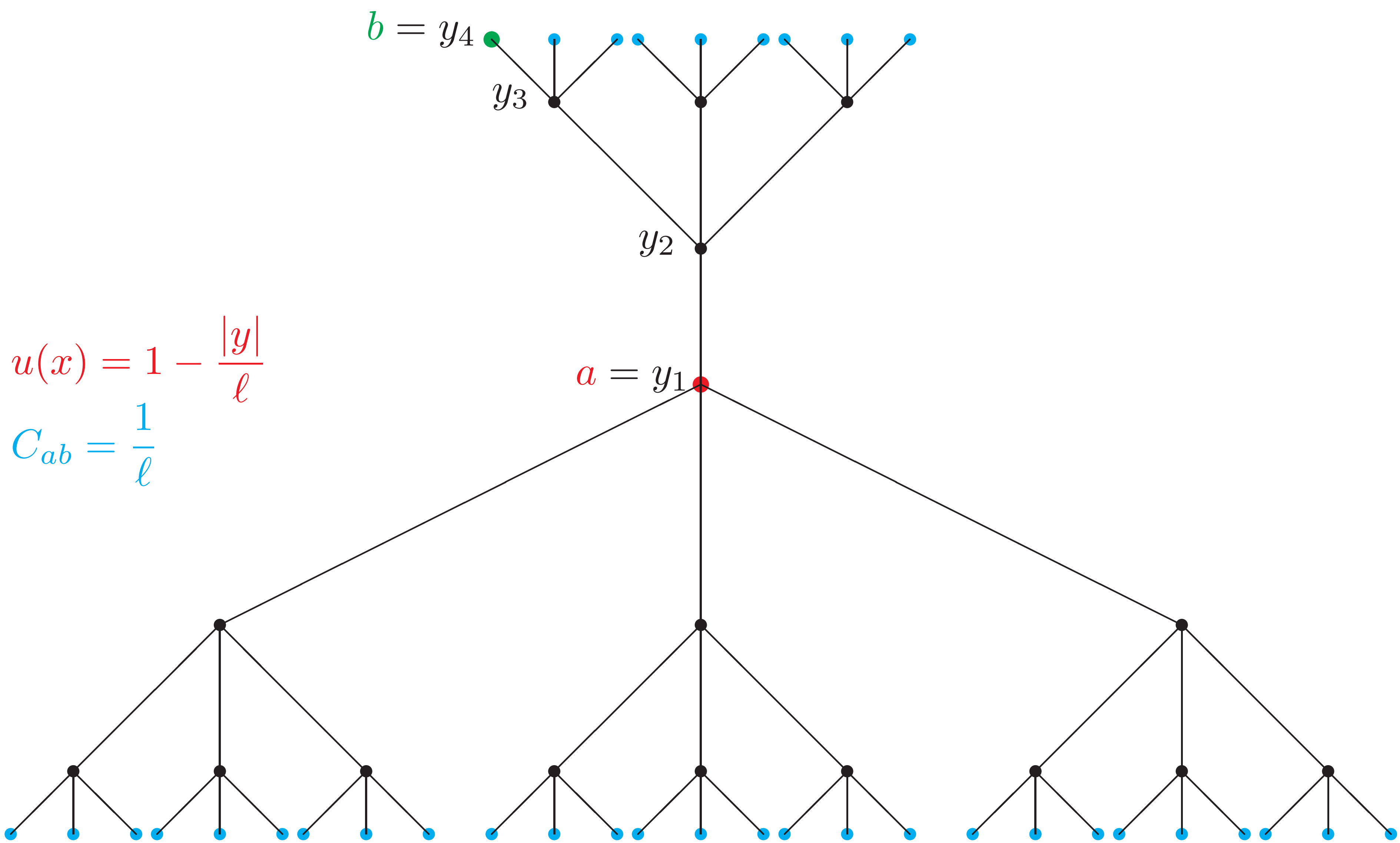
Computing the equilibrium measures $\nu^{F\cup H}$



$$q(i,j) = \frac{k-1}{\ell(k-2)^2}(i-1) - \frac{j-1}{k-2} +$$

$$\frac{1}{\ell(k-2)^2}\Big((\ell - i + 1)(k-1)^{\ell+1} - \ell(k-1)^{\ell+1-(i-1)-(j-1)}\Big)$$

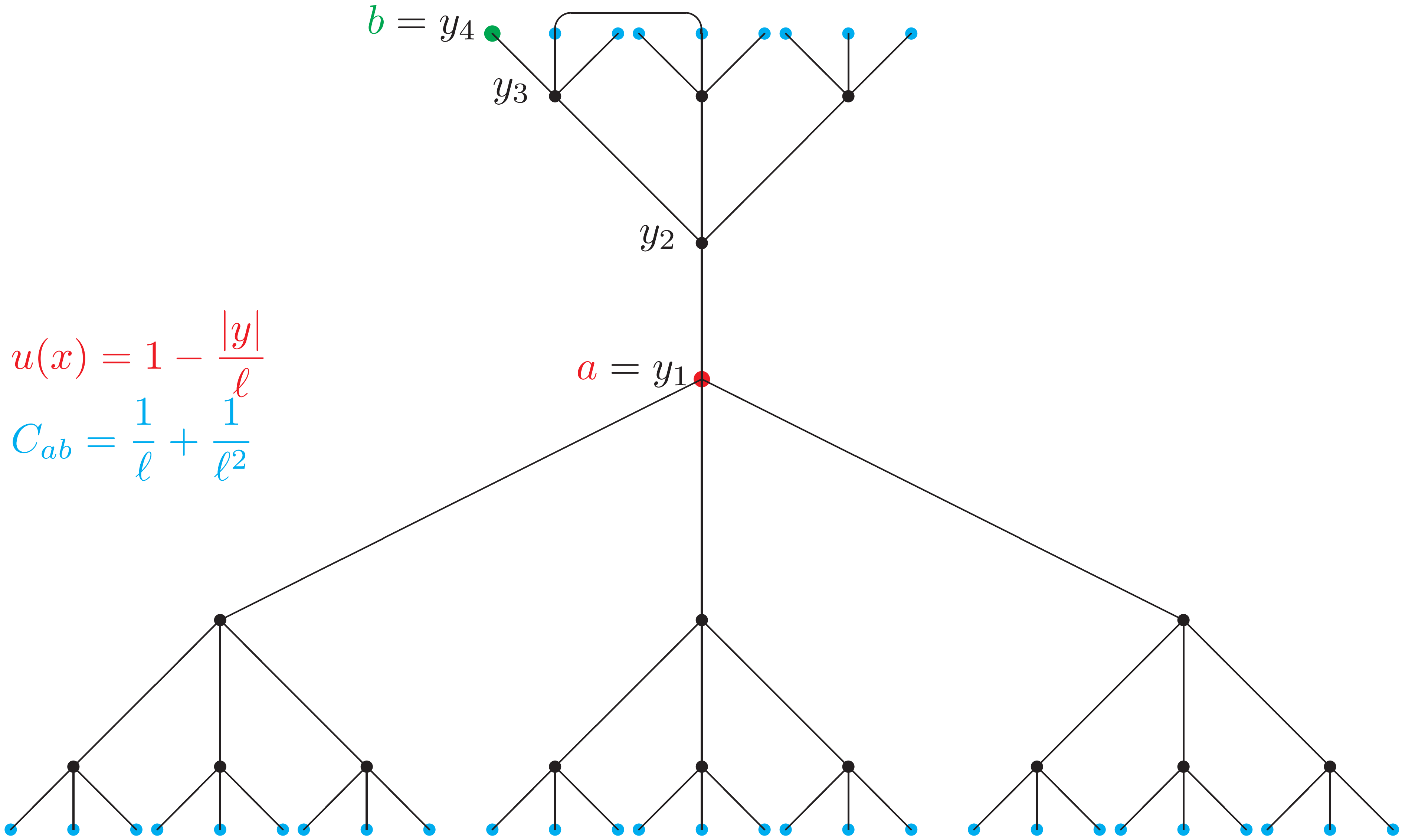
The Effective Resistance of a Tree: T_ℓ^k



$$u(x) = 1 - \frac{|y|}{\ell}$$

$$C_{ab} = \frac{1}{\ell}$$

Some More Edges



$$u(x) = 1 - \frac{|y|}{\ell}$$

$$C_{ab} = \frac{1}{\ell} + \frac{1}{\ell^2}$$