Cheeger constant and Eigenvalues on Networks

Enrique Bendito, Ángeles Carmona*, Andrés M. Encinas and José M. Gesto

Departament de Matemàtica Aplicada III
Universitat Politècnica de Catalunya.
08034 Barcelona.
Spain.

The classical isoperimetric problem establishes that, among all the subsets of $\mathbb{R}^3$ with given volume, the sphere has the minimum area and a more modern formulation concerns compact n-dimensional manifolds. In 1970 J. Cheeger proved that the first eigenvalue of the Laplacian of a manifold can be bounded below in terms of the infimum of the ratios between area and volume, now called Cheeger constant. In 1982 P. Buser established that the eigenvalue spectrum of the manifold could also be bounded above in terms of the Cheeger constant. Buser also introduced a combinatorial analog of the Cheeger constant that applies to finite, simple networks since they can be considered as the discrete version of compact Riemannian manifolds. So, it is not surprising that the Cheeger constant is related with the eigenvalues of the network. The most celebrated inequalities are

$$\frac{h(\Gamma)^2}{2} \leq \lambda(\Gamma) \leq 2h(\Gamma).$$

It is known that obtaining the Cheeger constant of a network is a NP-complete problem. Therefore any non trivial bound of it has interest and it is potentially important. In the last years, bounds involving only relevant parameters of the network such as size, order, diameter or degrees have been obtained.

The techniques used here are the usual in this context, that is to apply to a particular function a discrete version of Green's Identity and the variational characterization of eigenvalues. The novelty lies in the functions we will consider. If one thinks about what functions are naturally associated with an arbitrary set in a general network, the only possible candidates seem to be the Dirac's measures and the characteristic function of the set. But they only express topological properties, *i.e.* if a vertex is in or out of the set, and they say nothing about the connectivity between vertices of the set. Hereby, if we try to consider functions that should take into account both aspects, topological and metrical, the natural choice is not other one that the equilibrium measure of the set. We will made the efficacy of this choice clear throughout some examples. Moreover we will pay special attention on distance-regular graphs, since in this type of graphs the equilibrium measures can be computed by hand.

For any $F \subset V$ proper subset, the main isoperimetric inequalities we obtain are

$$\mathsf{vol}_c(\partial F)\gamma_m^F \leq \mathsf{vol}_\nu(F) \leq \mathsf{vol}_c(\partial F)\gamma_M^F$$

where $\gamma_M^F = \max\Big\{\gamma^F(x): \; x \in \delta(F^c)\Big\}$ and $\gamma_m^F = \min\Big\{\gamma^F(x): \; x \in \delta(F^c)\Big\}$ and where γ^F is the equilibrium measure of F. We also show the sharpness of these bounds throughout the analysis of some well-known examples as complete graphs, cycles, paths, cubes and homogeneous trees.

The second part of this work is devoted to obtain lower and upper bounds on the first non-null eigenvalue for self-adjoint boundary value problems related with the Laplace-Beltrami operator on a network. So, for the first Dirichlet eigenvalue of a proper subset F, we obtain

$$\min_{x \in F}\left\{\frac{1}{\gamma^F(x)}\right\} \leq \lambda_{_d}(F) \leq \frac{||\gamma^F||_{1,\nu}}{||\gamma^F||_{2,\nu}^2}$$

and analogous bounds are given for Poisson, Neumann and mixed Dirchlet-Neumann problems.

Acknowledgment

This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología), under Project BFM2003-06014 and the ETSECCPB.