The M–matrix Moore–Penrose inverse problem for weighted paths

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Abstract. A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative. However, when the matrix is an irreducible and singular M–matrix it is known that it has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. We focus here in characterizing when the Moore–Penrose inverse of a symmetric, singular, irreducible and tridiagonal M–matrix is itself a M–matrix.

Key words: M–matrix, Moore–Penrose inverse, Laplacian, tridiagonal–matrix.

1 Statement of the Problem

The matrices that can be expressed as $L = kI - A$, where $k > 0$ and $A \geq 0$, appear in relation with systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics. Of course, the combinatorial community can recognize within this type of matrices, the combinatorial Laplacian of a k–regular graph where A is the adjacency matrix.

If k is at least the spectral radio of A, then L is called an *M–matrix*. We remark that M–matrices arise naturally in some discretizations of differential operators, particularly those with a minimum/maximum principle, such as the Laplacian, and as such are well–studied in scientific computing. In fact, M–matrices satisfy monotonicity properties that are the discrete counterpart of the minimum principle, and this makes them suitable for the resolution of large sparse systems of linear equations by iterative methods.

A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative, [3]. However, when the matrix is an irreducible

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and singular M–matrix it is known that it has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. For instance, it may happens that the Moore–Penrose inverse has some negative entries. We focus here at characterizing when the Moore–Penrose inverse of a symmetric, singular, irreducible and tridiagonal M–matrix is itself a M–matrix. This problem has been widely studied for several types of matrices, see for instance [2,5,6,7].

2 Preliminaries

Given $c_1, \ldots, c_{n-1} > 0$ and $d_1, \ldots, d_n \geq 0$ such that the tridiagonal matrix

$$\mathsf{M} = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-1} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix} \tag{1}$$

is a singular M–matrix, we aim here at determining when its Moore–Penrose inverse $\mathsf{M}^\dagger$ is also a M–matrix. Although this problem lies in the framework of linear algebra, we have tackled it by applying methods from the Operator Theory on finite networks, see [2]. To do this we take into account that the off–diagonal entries of M can be identified with the conductance function of a weighted n–path. Specifically, if $V = \{x_1, \ldots, x_n\}$, then we can consider the weighted path $\Gamma = (V, c)$ where the *conductance* between vertices x_i and x_{i+1} is defined by $c(x_i, x_{i+1}) = c_i$.

Each real function on V can be identified with a (column) vector of $\mathbb{R}^n$ and hence each endomorphism of the space of real functions on V can be identified with a matrix of order n and conversely. In particular, the matrix obtained by choosing $d_1 = c_1$, $d_n = c_{n-1}$ and $d_i = c_{i-1} + c_i$ for $i = 2, \ldots, n-1$ is nothing but the combinatorial Laplacian of the network Γ. Therefore, M can be considered as a *perturbed Laplacian of* Γ in the sense of [1] and then we ask which perturbed Laplacians of Γ are singular and positive semi–definite.

From the Operator Theory point of view, the perturbed Laplacians are identified with the so–called *discrete Schrödinger operators of* Γ, see for instance [4] and references therein for several physical interpretations. In addition, this terminology suggests some sort of relationship with the differential operators with the same name. In fact, many of the techniques and results in this framework appear as the discrete counterpart of the standard treatment of the resolvent of elliptic operators on Riemannian manifolds.

3 Symmetric tridiagonal M–matrices

It was proved in [2] that the matrix given in (1) is a singular M–matrix iff there exists $\omega_1, \ldots, \omega_n > 0$ such that $\omega_1^2 + \ldots + \omega_n^2 = 1$ and

$$d_1 = \frac{c_1\omega_2}{\omega_1}, \quad d_n = \frac{c_{n-1}\omega_{n-1}}{\omega_n} \quad \text{and} \quad d_j = \frac{1}{\omega_j}(c_j\omega_{j+1} + c_{j-1}\omega_{j-1}) \tag{2}$$

for any $j = 2, \ldots, n-1$.

In the sequel any $\omega = (\omega_1, \ldots, \omega_n)^T \in \mathbb{R}^n$ such that $\omega_1, \ldots, \omega_n > 0$ and $\omega_1^2 + \ldots + \omega_n^2 = 1$, is called *weight*. Moreover, the matrix given in (1) is denoted by $\mathsf{M}(\mathsf{c}, \omega)$, where $\mathsf{c} = (c_1, \ldots, c_{n-1})^T$ and the diagonal entries are given by (2). We drop ω in all the expressions when ω is constant; that is when $\omega_j = \dfrac{1}{\sqrt{n}}$, for any $j = 1, \ldots, n$.

Throughout the paper, we use the conventions $\sum\limits_{l=i}^{j} a_l = 0$ and $\prod\limits_{l=i}^{j} a_l = 1$ when $j < i$.

Proposition 1 ([2, Corollary 5.2]). *The Moore–Penrose inverse of* $\mathsf{M}(\mathsf{c}, \omega)$ *is* $\mathsf{M}^\dagger(\mathsf{c}, \omega) = (g_{ij})$*, where*

$$g_{ji} = g_{ij} = \omega_i\omega_j \left[\sum_{k=1}^{i-1} \frac{\left(\sum\limits_{l=1}^{k} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} + \sum_{k=i}^{n-1} \frac{\left(\sum\limits_{l=k+1}^{n} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} - \sum_{k=i}^{j-1} \frac{\left(\sum\limits_{l=k+1}^{n} \omega_l^2\right)}{c_k\omega_k\omega_{k+1}} \right]$$

for any $1 \le i \le j \le n$.

If we take into account that the Moore–Penrose inverse of a symmetric and positive semi–definite matrix is itself symmetric and positive semi–definite, as a by–product of the expression of $\mathsf{M}^\dagger(\mathsf{c}, \omega)$ we can easily characterize when it is a M–matrix.

Theorem 1. $\mathsf{M}^\dagger(\mathsf{c}, \omega)$ *is a M–matrix iff* $g_{ii+1} \le 0$ *for any* $i = 1, \ldots, n-1$*, that is; iff*

$$\frac{\left(\sum\limits_{l=i+1}^{n} \omega_l^2\right)\left(\sum\limits_{l=1}^{i} \omega_l^2\right)}{c_i\omega_i\omega_{i+1}} \ge \sum_{k=1}^{i-1} \frac{\left(\sum\limits_{l=1}^{k} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} + \sum_{k=i+1}^{n-1} \frac{\left(\sum\limits_{l=k+1}^{n} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}}, \quad i = 1, \ldots, n-1.$$

The Moore–Penrose inverse for the normalized Laplacian; that is, when ω is the square root of the generalized degree, was obtained in [8, Theorem 9]. In addition, the conclusion of the above Theorem for ω constant was given in [5, Lemma 3.1].

Theorem 1 also implies that when $n = 2$, then $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is always a M–matrix. In fact, for any $c > 0$ and any $0 < x < 1$, if $\omega = (x, \sqrt{1-x^2})^T$, we get $\mathsf{M}(\mathsf{c},\omega) = c \begin{bmatrix} \frac{\sqrt{1-x^2}}{x} & -1 \\ -1 & \frac{x}{\sqrt{1-x^2}} \end{bmatrix}$ and

$$\mathsf{M}^\dagger(\mathsf{c},\omega) = \frac{x\sqrt{1-x^2}}{c} \begin{bmatrix} 1-x^2 & -x\sqrt{1-x^2} \\ -x\sqrt{1-x^2} & \omega^2 \end{bmatrix}.$$

Corollary 1. *When $n = 3$, $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is a M–matrix iff*

$$\frac{\omega_1^3}{\omega_3(1-\omega_3^2)} \le \frac{c_1}{c_2} \le \frac{\omega_1(1-\omega_1^2)}{\omega_3^3}.$$

On the other hand, for any conductance c there exists a weight ω such that $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is a M–matrix. In particular, if ω is constant, then $\mathsf{M}^\dagger(\mathsf{c})$ is a M–matrix iff $\frac{1}{2} \le \frac{c_1}{c_2} \le 2$.

Proof. In this case the system of inequalities in Corollary 1 is reduced to

$$\frac{\omega_1^2(1-\omega_1^2)}{c_1\omega_1\omega_2} \ge \frac{\omega_3^4}{c_2\omega_2\omega_3} \quad \text{and} \quad \frac{\omega_3^2(1-\omega_3^2)}{c_2\omega_2\omega_3} \ge \frac{\omega_1^4}{c_1\omega_1\omega_2},$$

that is equivalently to the claimed inequalities. Finally, given $c_1, c_2 > 0$, consider

$$0 < x < \min\left\{\sqrt{\frac{c_1}{c_1+c_2}}, \sqrt{\frac{c_2}{c_1+c_2}}\right\}$$

and choose $\omega_1 = \omega_3 = x$ and $\omega_2 = \sqrt{1-2x^2}$. Then

$$\frac{\omega_1^3}{\omega_3(1-\omega_3^2)} = \frac{x^2}{1-x^2} \le \frac{c_1}{c_2} \le \frac{1-x^2}{x^2} = \frac{\omega_1(1-\omega_1^2)}{\omega_3^3}$$

and hence, $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is a M–matrix. □

Corollary 2. *When the weight is constant, then $\mathsf{M}^\dagger(\mathsf{c})$ is a M–matrix iff $n \le 4$ and moreover either $\frac{1}{2} \le \frac{c_1}{c_2} \le 2$ when $n = 3$ or $c_1 = c_3$ and $c_2 = 2c_1$ when $n = 4$.*

Proof. When ω is constant, the system of inequalities in Corollary 1 is equivalent to

$$\frac{i(n-i)}{c_i} \ge \sum_{k=1}^{i-1} \frac{k^2}{c_k} + \sum_{k=i+1}^{n-1} \frac{(n-k)^2}{c_k}, \qquad i = 1, \dots, n-1.$$

Expanding the above inequalities up to $i = 3$, we get that

$$\frac{(n-1)}{c_1} \geq \sum_{k=2}^{n-1} \frac{(n-k)^2}{c_k},$$
$$\frac{2(n-2)}{c_2} \geq \frac{1}{c_1} + \sum_{k=3}^{n-1} \frac{(n-k)^2}{c_k},$$
$$\frac{3(n-3)}{c_3} \geq \frac{1}{c_1} + \frac{4}{c_2} + \sum_{k=4}^{n-1} \frac{(n-k)^2}{c_k},$$

and hence that

$$\frac{(n-1)}{c_1} \geq \sum_{k=2}^{n-1} \frac{(n-k)^2}{c_k},$$
$$\frac{(n-2)}{c_2} \geq \sum_{k=3}^{n-1} \frac{(n-k)^2}{c_k},$$
$$\frac{2(n-3)}{c_3} \geq \frac{n}{c_2} + \sum_{k=4}^{n-1} \frac{(n-k)^2}{c_k},$$

which implies that

$$\frac{(n-3)(5n-4-n^2)}{c_3} \geq 2(n-1) \sum_{k=4}^{n-1} \frac{(n-k)^2}{c_k}.$$

Therefore, $5n - n^2 - 4$ must be non–negative and this occurs iff $n \leq 4$. Moreover, when $n = 4$, the system of inequalities in Theorem 1 is

$$\frac{3}{c_1} \geq \frac{4}{c_2} + \frac{1}{c_3}, \quad \frac{4}{c_2} \geq \frac{1}{c_1} + \frac{1}{c_3} \quad \text{and} \quad \frac{3}{c_3} \geq \frac{1}{c_1} + \frac{4}{c_2},$$

which implies that

$$\frac{1}{c_1} \geq \frac{1}{c_3}, \quad \frac{2}{c_2} \geq \frac{1}{c_3} \quad \text{and} \quad \frac{3}{c_3} \geq \frac{1}{c_1} + \frac{4}{c_2} \geq \frac{1}{c_1} + \frac{2}{c_3} \geq \frac{3}{c_3}$$

and hence $c_2 = 2c_3$ and $c_1 = c_3$. Conversely, when ω is constant and $c_1 = c_3$ and $c_2 = 2c_1$, then system of inequalities in Theorem 1 is satisfied, and hence $\mathsf{M}^\dagger(\mathsf{c})$ is a M–matrix. □

The above result was also obtained in [5] by using a different approach.

The cases ω constant and $n = 2, 3$ are the only ones in which we tackle directly the system of inequalities in Theorem 1. For $n \geq 4$ we will follow a different way that also works for $n = 2, 3$. In the sequel, for a given weight ω, the set of conductances c such that $\mathsf{M}^\dagger(\mathsf{c}, \omega)$ is an M–matrix is denoted by $\mathsf{C}(\omega)$.

For sake of simplicity for any $j = 1, \dots, n-1$ we define $W_j = \sum_{l=1}^{j} \omega_l^2$. Moreover, given $n \geq 2$ and a weight ω, we consider the following irreducible $(n-1)$–order Z–matrix.

$$\mathsf{A}(\omega) = \begin{bmatrix} \frac{W_1(1-W_1)}{\omega_1\omega_2} & -\frac{(1-W_2)^2}{\omega_2\omega_3} & -\frac{(1-W_3)^2}{\omega_3\omega_4} & \cdots & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n} \\ -\frac{W_1^2}{\omega_1\omega_2} & \frac{W_2(1-W_2)}{\omega_2\omega_3} & -\frac{(1-W_3)^2}{\omega_3\omega_4} & \cdots & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\frac{W_1^2}{\omega_1\omega_2} & -\frac{W_2^2}{\omega_2\omega_3} & \cdots & \frac{W_{n-2}(1-W_{n-2})}{\omega_{n-2}\omega_{n-1}} & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n} \\ -\frac{W_1^2}{\omega_1\omega_2} & -\frac{W_2^2}{\omega_2\omega_3} & \cdots & -\frac{W_{n-2}^2}{\omega_{n-2}\omega_{n-1}} & \frac{W_{n-1}(1-W_{n-1})}{\omega_{n-1}\omega_n} \end{bmatrix}.$$

If for a conductance c, we define $\mathsf{c}^{-1} = (c_1^{-1}, \dots, c_{n-1}^{-1})^T$, then from Theorem 1, $\mathsf{M}^\dagger(\mathsf{c}, \omega)$ is a M–matrix iff all the entries of the vector $\mathsf{A}(\omega)\mathsf{c}^{-1}$ are non–negative. Therefore, by applying [3, Exercise 6.4.14], if $\mathsf{C}(\omega) \neq \emptyset$ for a weight ω, then $\mathsf{A}(\omega)$ is a M–matrix. Conversely when $\mathsf{A}(\omega)$ is a non singular M–matrix then $\mathsf{c} \in \mathsf{C}(\omega)$ iff $\mathsf{c}^{-1} = \mathsf{A}^{-1}(\omega)\mathsf{a}$, where a in non null and all its entries are nonnegative. In the next result we characterize when $\mathsf{A}(\omega)$ is a M–matrix.

Lemma 1. *For any $n \geq 2$ and any weight ω,* $\operatorname{rank} \mathsf{A}(\omega) \geq n-2$ *and moreover*

$$\det \mathsf{A}(\omega) = \omega_1\omega_n \Big(\prod_{k=1}^{n-1} \omega_k^2\Big)^{-1} \Big(\prod_{k=1}^{n-2} W_k\Big) \left[\omega_{n-1}^2 - \sum_{j=1}^{n-3} W_j \prod_{k=j+1}^{n-2} \frac{(1-W_k)}{W_k}\right].$$

In addition, $\mathsf{A}(\omega)$ is a M–matrix iff

$$\omega_{n-1}^2 \geq \sum_{j=1}^{n-3} W_j \prod_{k=j+1}^{n-2} \frac{(1-W_k)}{W_k}$$

and it is singular when the equality holds.

Proof. To prove the claims it suffices to apply the Gauss Method. The first step consists in substracting the $(i+1)$–row to the i–row, for any $i = 1, \dots, n-2$.

Secondly, we add to the last row the i–row multiplied by W_i, for any $i = 1, \dots, n-2$ and the third step consists in dividing each row, except the last one, by its diagonal entry. So, if we consider the matrix

$$\mathsf{P}(\omega) = \begin{bmatrix} \frac{\omega_1\omega_2}{W_1} & -\frac{\omega_1\omega_2}{W_1} & 0 & \cdots & 0 \\ 0 & \frac{\omega_2\omega_3}{W_2} & -\frac{\omega_2\omega_3}{W_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} & -\frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} \\ \omega_1^2 & \omega_2^2 & \cdots & \omega_{n-2}^2 & \omega_{n-1}^2 + \omega_n^2 \end{bmatrix},$$

then, $\det \mathsf{P}(\omega) = \Big(\prod_{k=1}^{n-2} \frac{\omega_k\omega_{k+1}}{W_k}\Big)\Big(\sum_{l=1}^{n} \omega_l^2\Big) = \frac{\omega_{n-1}}{\omega_1}\Big(\prod_{k=1}^{n-2} \omega_k^2\Big)\Big(\prod_{k=1}^{n-2} W_k\Big)^{-1}$ and moreover

$$\mathsf{P}(\omega)\mathsf{A}(\omega)=\begin{bmatrix} 1 & -\frac{\omega_1(1-W_2)}{\omega_3 W_1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\frac{\omega_2(1-W_3)}{\omega_4 W_2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\frac{\omega_3(1-W_4)}{\omega_5 W_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -\frac{\omega_{n-2}(1-W_{n-1})}{\omega_n W_{n-2}} \\ 0 & -\frac{W_1(1-W_2)}{\omega_2\omega_3} & -\frac{W_2(1-W_3)}{\omega_3\omega_4} & \cdots & -\frac{W_{n-3}(1-W_{n-2})}{\omega_{n-2}\omega_{n-1}} & \omega_{n-1}\omega_n \end{bmatrix}$$

Now, for any $i = 2, \ldots, n-2$ we define

$$\gamma_i = \frac{\sum_{j=1}^{i-1} W_j\Big(\prod_{k=1}^{j} W_k\Big)\Big(\prod_{k=j+1}^{i} (1-W_k)\Big)}{\omega_i\omega_{i+1}\prod_{k=1}^{i-1} W_k}.$$

Then, $\gamma_2 = \dfrac{W_1(1-W_2)}{\omega_2\omega_3}$ and moreover

$$\gamma_i\,\frac{\omega_i(1-W_{i+1})}{\omega_{i+2}W_i} + \frac{W_i(1-W_{i+1})}{\omega_{i+1}\omega_{i+2}} = \gamma_{i+1},$$

for any $i = 2, \ldots, n-3$. Therefore, if $\gamma = (0, \gamma_2, \ldots, \gamma_{n-2})$ and

$$\mathsf{Q}(\omega) = \begin{bmatrix} \mathrm{I} & 0 \\ \gamma & 1 \end{bmatrix}\mathsf{P}(\omega) = \begin{bmatrix} \frac{\omega_1\omega_2}{W_1} & -\frac{\omega_1\omega_2}{W_1} & 0 & \cdots & 0 \\ 0 & \frac{\omega_2\omega_3}{W_2} & -\frac{\omega_2\omega_3}{W_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} & -\frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} \\ q_1(\omega) & q_2(\omega) & \cdots & q_{n-2}(\omega) & q_{n-1}(\omega) \end{bmatrix},$$

where I is de Identity matrix of order $n-2$, and

$$\begin{aligned} q_1(\omega) &= \omega_1^2, \\ q_2(\omega) &= \omega_2^2 + \gamma_2\frac{\omega_2\omega_3}{W_2}, \\ q_j(\omega) &= \omega_j^2 + \gamma_j\frac{\omega_j\omega_{j+1}}{W_j} - \gamma_{j-1}\frac{\omega_{j-1}\omega_j}{W_{j-1}}, \quad j = 3, \ldots, n-2, \\ q_{n-1}(\omega) &= \omega_{n-1}^2 + \omega_n^2 - \gamma_{n-2}\frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}}. \end{aligned}$$

we get that $\det \mathsf{Q}(\omega) = \det \mathsf{P}(\omega)$ and moreover $\mathsf{Q}(\omega)\mathsf{A}(\omega) = \mathsf{B}(\omega)$ where

$$\mathsf{B}(\omega) = \begin{bmatrix} 1 - \frac{\omega_1(1-W_2)}{\omega_3 W_1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\frac{\omega_2(1-W_3)}{\omega_4 W_2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\frac{\omega_3(1-W_4)}{\omega_5 W_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -\frac{\omega_{n-2}(1-W_{n-1})}{\omega_n W_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \omega_n \left[\omega_{n-1} - \gamma_{n-2}\frac{\omega_{n-2}}{W_{n-2}}\right] \end{bmatrix}$$

and hence $\operatorname{rank} \mathsf{A}(\omega) \geq n-2$ and $\det \mathsf{A}(\omega) = \dfrac{\omega_n}{\det \mathsf{P}(\omega)}\left[\omega_{n-1} - \gamma_{n-2}\dfrac{\omega_{n-2}}{W_{n-2}}\right]$. The expression for $\det \mathsf{A}(\omega)$ follows by substituting γ_{n-2} by its value in the above equality. Finally, from [3, Theorems 6.2.4 and 6.4.16], we know that $\mathsf{A}(\omega)$ is a M–matrix iff all its principal minors are non–negative. From the expression of $\mathsf{Q}(\omega)\mathsf{A}(\omega)$, this occurs iff $\omega_{n-1} \geq \gamma_{n-2}\dfrac{\omega_{n-2}}{W_{n-2}}$, since all principal minors of $\mathsf{Q}(\omega)$ are positive. □

The above formulae for $\det\mathsf{A}(\omega)$ implies that $\det \mathsf{A}(\omega) = \omega_1\omega_2$ when $n = 2$, that $\det \mathsf{A}(\omega) = \omega_1\omega_3$ when $n = 3$, whereas $\det \mathsf{A}(\omega) = \dfrac{\omega_1\omega_4}{\omega_2^2\omega_3^2}\left(\omega_3^2\omega_2^2 - \omega_1^2\omega_4^2\right)$ when $n = 4$. So, when $n = 2, 3$ then $\mathsf{A}(\omega)$ is always a nonsingular M–matrix, whereas when $n = 4$ then $\mathsf{A}(\omega)$ is a M–matrix iff $\omega_2\omega_3 \geq \omega_1\omega_4$ and singular when, in the equality holds. On the other hand, when ω is constant

$$\det \mathsf{A} = \frac{(n-2)!}{n}\left[1 - (n-1)\sum_{j=1}^{n-3} j \binom{n-1}{j}^{-1}\right]$$

and hence A is a M–matrix iff $n \leq 4$, since $1-(n-1)(n-3)\binom{n-1}{n-3}^{-1} = \dfrac{4-n}{n-2}$. This explains why $\mathsf{C} = \emptyset$ when $n > 5$, as was proved in Corollary 2.

Consider $D\colon \Omega(V) \longrightarrow \mathbb{R}$ defined as $D(\omega) = \omega_{n-1}^2 - \sum\limits_{j=1}^{n-3} W_j \prod\limits_{k=j+1}^{n-2} \frac{(1-W_k)}{W_k}$. From the above reasoning it is clear that $\mathsf{C}(\omega) = \emptyset$ when $D(\omega) < 0$.

Proposition 2. *If $D(\omega) > 0$ then $\mathsf{C}(\omega) = \left\{c(1, \widehat{c}_2, \ldots, \widehat{c}_{n-1})^T : c > 0\right\}$, where*

$$\widehat{c}_j = \frac{\omega_1\omega_2 \prod\limits_{k=2}^{j}(1-W_k)}{\omega_j\omega_{j+1}\prod\limits_{k=1}^{j-1} W_k}, \qquad j = 2, \ldots, n-1.$$

Proof. From Lemma 1, $\mathsf{A}(\omega)$ is a singular M–matrix whose rank equals $n-2$ and hence [3, Theorem 6.4.16] implies that $\mathsf{c} \in \mathsf{C}(\omega)$ iff $\mathsf{c}^{-1} \in \ker \mathsf{A}(\omega)$. Therefore, with the notations of Lemma 1, $\mathsf{c} \in \mathsf{C}(\omega)$ iff $\mathsf{c}^{-1} \in \ker \mathsf{B}(\omega)$ and the result follows. □

Proposition 3. *If $D(\omega) > 0$, then $\mathsf{c} \in \mathsf{C}(\omega)$ iff*

$$c_i = \left(\sum_{j=1}^{n-1} \alpha_{ij}(\omega) a_j \right)^{-1}, \quad i = 1, \ldots, n-1,$$

where

$$\alpha_{ij}(\omega) = q_j(\omega) \frac{\omega_{n-1}}{\omega_n D(\omega)} \prod_{l=i}^{n-2} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l},$$

if $1 \leq j < i \leq n-1$,

$$\alpha_{jj}(\omega) = \frac{\omega_j \omega_{j+1}}{W_j} + q_j(\omega) \frac{\omega_{n-1}}{\omega_n D(\omega)} \prod_{l=j}^{n-2} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l},$$

if $j = 1, \ldots, n-2$,

$$\begin{aligned} \alpha_{ij}(\omega) = & -\frac{\omega_{j-1}\omega_j}{W_{j-1}} \prod_{l=i}^{j-2} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l} + \frac{\omega_j \omega_{j+1}}{W_j} \prod_{l=i}^{j-1} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l} \\ & + q_j(\omega) \frac{\omega_{n-1}}{\omega_n D(\omega)} \prod_{l=i}^{n-2} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l}, \end{aligned}$$

if $1 \leq i < j \leq n-2$,

$$\begin{aligned} \alpha_{in-1}(\omega) = & -\frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} \prod_{l=i}^{n-3} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l} \\ & + q_{n-1}(\omega) \frac{\omega_{n-1}}{\omega_n D(\omega)} \prod_{l=i}^{n-2} \frac{\omega_l (1 - W_{l+1})}{\omega_{l+2} W_l}, \end{aligned}$$

if $i = 1, \ldots, n-2$,

$$\alpha_{n-1n-1}(\omega) = q_{n-1}(\omega) \frac{\omega_{n-1}}{\omega_n D(\omega)}$$

and $a_1, \ldots, a_{n-1} \geq 0$ and $a_1 + \cdots + a_{n-1} > 0$.

Proof. From Lemma 1 we know that $\mathsf{A}(\omega)$ is a non–singular M–matrix. In addition, with the notations of Lemma 1 we get that $\mathsf{A}^{-1}(\omega) = \mathsf{B}^{-1}(\omega)\mathsf{Q}(\omega)$. Moreover,

$$\mathsf{B}^{-1}(\omega)=\begin{bmatrix} 1 & \frac{\omega_1(1-W_2)}{\omega_3 W_1} & \prod_{l=1}^{2}\frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} & \prod_{l=1}^{3}\frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} & \cdots & \frac{\omega_{n-1}}{\omega_n D(\omega)}\prod_{l=1}^{n-2}\frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} \\ 0 & 1 & \frac{\omega_2(1-W_3)}{\omega_4 W_2} & \prod_{l=2}^{3}\frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} & \cdots & \frac{\omega_{n-1}}{\omega_n D(\omega)}\prod_{l=2}^{n-2}\frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} \\ 0 & 0 & 1 & \frac{\omega_3(1-W_4)}{\omega_5 W_3} & \cdots & \frac{\omega_{n-1}}{\omega_n D(\omega)}\prod_{l=3}^{n-2}\frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \frac{\omega_{n-1}}{\omega_n D(\omega)}\frac{\omega_{n-2}(1-W_{n-1})}{\omega_n W_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\omega_{n-1}}{\omega_n D(\omega)} \end{bmatrix}.$$

The proof ends taking into account that $\mathsf{A}^{-1}(\omega)=\big(\alpha_{ij}(\omega)\big)$. □

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