On the M–property for distance–regular graphs

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M-matrix

Matrices with **non–positive** off–diagonal and **non–negative** diagonal entries

$$\boxed{L = kI - A}$$

$k > 0, A \geq 0$ and the diagonal entries of A are less or equal to k

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If $k \geq \rho(A)$, then L is called an **M–matrix**

Where can we find M-matrices?

- Finite difference methods for solving PDE
- Growth models in economics
- Markov processes in probability and statistics

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Symmetric and irreducible M–matrices

- **non–singular** --→ discrete Dirichlet problem --→ its inverse corresponds with the Green operator associated with the boundary value problem
- **singular** --→ discrete Poisson equation --→ its Moore–Penrose inverse corresponds with the Green operator too

Generalized inverses. The Moore–Penrose inverse

A **generalized inverse** of matrix A

- exists for a class of matrices larger than the class of non–singular matrices
- has some of the properties of the usual inverse
- reduces to the usual inverse when A is non–singular

For every finite matrix A there is a unique matrix X satisfying the Penrose equations

$$\mathsf{A}X\mathsf{A} = \mathsf{A} \qquad (\mathsf{A}X)^* = \mathsf{A}X$$
$$X\mathsf{A}X = X \qquad (X\mathsf{A})^* = X\mathsf{A}$$

X is the **Moore–Penrose inverse**, and it is denoted by $\mathsf{A}^\dagger$

Statement of the Problem

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Given an M–matrix, there exists a generalized inverse which is also an M–**matrix**?

The Combinatorial Laplacian

When the Moore–Penrose inverse of the combinatorial Laplacian of a graph is an M-matrix?

$\mathsf{L}^{\dagger}$ is an M–matrix iff $\mathsf{L}^{\dagger}{}_{ij} \leq 0$ for any $x_i, x_j \in V$ with $i \neq j$

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Example: Petersen

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Example: Cycle

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When the Moore–Penrose inverse of the combinatorial Laplacian of a graph is the combinatorial Laplacian of a network?

Network

- **Finite network**: $\Gamma = (V, E, c)$
- The **conductance**: $c\colon V \times V \longrightarrow [0, +\infty)$ such that

$$\begin{cases} c(x,x) = 0 & \text{for any } x \in V \\ c(x,y) > 0 & \text{for } x \sim y. \end{cases}$$

- $\mathcal{C}(\mathbf{V})$ is the set of real–valued functions on V
  - the functions in $\mathcal{C}(V) \dashrightarrow \mathbb{R}^{|V|}$
  - the endomorphisms of $\mathcal{C}(V) \dashrightarrow |V|$–order square matrices

Network

The **combinatorial Laplacian** of Γ

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\Big(u(x) - u(y)\Big), \quad x \in V$$

- $\mathcal{L}$ is a positive semi–definite self–adjoint operator
- $\mathcal{L}$ has 0 as its lowest eigenvalue whose associated eigenfunctions are constant
- $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular M–matrix, $\mathbf{L}$

Definition

Γ has the *M–property* iff $\mathsf{L}^{\dagger}$ is a M–matrix

Distance–regular graphs

Let Γ be a distance–regular graph with intersection array

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, \ldots, c_D\}$$

Γ is regular of degree k, and

$$k = b_0, \quad b_D = c_0 = 0, \quad c_1 = 1, \quad a_i + b_i + c_i = k$$

Examples

n–Cycle: $\iota(C_n) = \left\{2, 1, \ldots, 1; 1, \ldots, 1, c_D\right\}$

The Heawood Graph:
$\iota(\Gamma) = \{3, 2, 2; 1, 1, 3\}$

The Petersen Graph:
$\iota(\Gamma) = \{3, 2; 1, 1\}$

The intersection array $\iota(\Gamma)$

Properties

i) $k_0 = 1$ and $k_i = \dfrac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i}, \quad i = 1, \ldots, D$

ii) $n = 1 + k + k_2 + \cdots + k_D$

iii) $k > b_1 \geq \cdots \geq b_{D-1} \geq 1$

iv) $1 \leq c_2 \leq \cdots \leq c_D \leq k$

v) If $i + j \leq D$, then $c_i \leq b_j$ and $k_i \leq k_j$ when, in addition, $i \leq j$

Proposition

Notation

- $a_1 = \lambda$
- $c_2 = \mu$

Bipartite and Antipodal distance–regular graphs Γ

(i) Γ is **bipartite** iff $a_i = 0,\ i = 1, \ldots, D$

(ii) Γ is **antipodal** iff $b_i = c_{D-i},\ i = 0, \ldots, D,\ i \neq \left\lfloor \frac{D}{2} \right\rfloor$

Distance–regular graphs with $k \geq 3$ other than bipartite and antipodal are **primitive**

The Moore–Penrose inverse of a distance–regular graph

Lemma

Let Γ be a distance–regular graph.

$$\mathsf{L}^{\dagger}{}_{ij} = \sum_{r=d(x_i,x_j)}^{D-1} \frac{1}{nk_r b_r}\Big(\sum_{l=r+1}^{D} k_l\Big) - \sum_{r=0}^{D-1} \frac{1}{n^2 k_r b_r}\Big(\sum_{l=0}^{r} k_l\Big)\Big(\sum_{l=r+1}^{D} k_l\Big)$$

for all $i, j = 1, \ldots, n$

The M–property

Proposition

A distance–regular graph Γ has the M–property iff

$$\sum_{j=1}^{D-1} \frac{1}{k_j b_j} \Big(\sum_{i=j+1}^{D} k_i \Big)^2 \leq \frac{n-1}{k}$$

Corollary

If Γ has the M–property and $D \geq 2$, then

$$\lambda \leq 3k - \frac{k^2}{n-1} - n$$

and hence $\mathbf{n < 3k}$.

Small diameter

Proposition

If Γ is a distance–regular graph with the M–property, then

$$D \leq 3$$

Proof.

If $D \geq 4$, then from property (v) of the parameters

$$3k < 1 + 3k \leq 1 + k + k_2 + k_3 \leq n$$

□

Diameter two

Proposition

A strongly regular graph with parameters (n, k, λ, μ) *has the* M*–property iff*

$$\mu \geq k - \frac{k^2}{n-1}$$

Observation

- Every antipodal strongly regular graph has the M–property
- Petersen graph, $(10, 3, 0, 1)$, does not have the M–property

Half of the strongly regular graphs have the M–property

Proposition

If Γ is a primitive strongly regular graph, then either Γ or $\overline{\Gamma}$ has the M–property.

The graphs Γ and $\overline{\Gamma}$, both of them, have the M–property iff Γ is a conference graph

Conference Graph
$(4m+1, 2m, m-1, m)$

Diameter three

Proposition

A distance–regular graph with $D = 3$ has the M–property iff

$$k^2 b_1 \left(b_2 c_2 + (b_2 + c_3)^2 \right) \leq c_2^2 c_3^2 (n-1)$$

Bipartite distance–regular graph with $D = 3$

$$\iota(\Gamma) = \{k, k-1, k-\mu; 1, \mu, k\}$$

Proposition

Γ *satisfies the M–property iff* $\frac{4k}{5} \leq \mu \leq k-1$

In particular, $k \geq 5$

Corollary

When, $1 \leq \mu < k-1$*, then either* Γ *or* Γ_3 *has the M–property, except when*

$$k-1 < 5\mu < 4k$$

in which case none of them has the M–property

Antipodal distance–regular graph

$$\iota(\Gamma) = \{k, t\mu, 1; 1, \mu, k\}$$

Proposition

An antipodal distance–regular graph with $D = 3$ has the M–property iff it is a Taylor graph $T(k, \mu)$ such that $k \geq 5$ and

$$\frac{k+3}{2} \leq \mu < k$$

Antipodal distance–regular graph

Corollary

If Γ is the Taylor graph $T(k,\mu)$ with $1 \leq \mu \leq k-2$, then

either Γ or Γ_2 has the M–property, except when

$\mu \in \{m-2, m-1, m, m+1\}$ *when* $k = 2m$ *and*

$\mu \in \{m-1, m, m+1\}$ *when* $k = 2m+1$*, in which case*

none of them has the M–property.

Bipartite and antipodal distance–regular graph with $D = 3$

$$\iota(\Gamma) = \{k, k-1, 1; 1, k-1, k\}$$

k–Crown graphs:

They are Taylor graphs with $\mu = k - 1$ and hence they

have the M–property iff $k \geq 5$

Primitive case

Lemma

$k_2 = k$ iff Γ is either C_6, C_7 or $T(\mu, k)$

Proposition

If Γ satisfies the M–property, then

$$1 < c_2 \leq b_1 < 2c_2 \qquad b_2 < c_3 \qquad k_3 \leq k - 3$$

Moreover, $k \geq 6$ and $c_2 < b_1$ when Γ is not a Taylor graph

Corollary

If Γ satisfies the M–property, then $b_2 < b_1 < k$ and $1 < c_2 < c_3$

Primitive case

Lemma

$k_2 = k$ iff Γ is either C_6, C_7 or $T(\mu, k)$

Proposition

If Γ satisfies the M–property, then

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Moreover, $k \geq 6$ and $c_2 < b_1$ when Γ is not a Taylor graph

Corollary

If Γ satisfies the M–property, then $b_2 < b_1 < k$ and $1 < c_2 < c_3$

Facts

- None of the *Shilla graphs* satisfy the M–property
- None of the families of primitive distance–regular graphs with diameter 3 satisfy the M–property

Conjecture

No primitive distance–regular graph with $D = 3$ satisfy the M–property.

Thank you!

The 5-Cycle has the M–property

The Moore–Penrose inverse of L is an M–matrix

$$\mathsf{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

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$$\mathsf{L}^{\dagger} = \begin{pmatrix} 2/5 & 0 & -1/5 & -1/5 & 0 \\ 0 & 2/5 & 0 & -1/5 & -1/5 \\ -1/5 & 0 & 2/5 & 0 & -1/5 \\ -1/5 & -1/5 & 0 & 2/5 & 0 \\ 0 & -1/5 & -1/5 & 0 & 2/5 \end{pmatrix}$$

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$$\mathsf{L}^{\dagger} = \begin{pmatrix} 2/5 & 0 & -1/5 & -1/5 & 0 \\ 0 & 2/5 & 0 & -1/5 & -1/5 \\ -1/5 & 0 & 2/5 & 0 & -1/5 \\ -1/5 & -1/5 & 0 & 2/5 & 0 \\ 0 & -1/5 & -1/5 & 0 & 2/5 \end{pmatrix}$$

Back

Petersen Graph does NOT have the M–property

The Moore–Penrose inverse of L is NOT an M–matrix

$$
\begin{aligned}
(\mathrm{L}^\dagger)_{ii} &= 0.33 \\
(\mathrm{L}^\dagger)_{ij} &= 0.03 && \text{if } d(x_i, x_j) = 1 \\
(\mathrm{L}^\dagger)_{ij} &= -0.07 && \text{if } d(x_i, x_j) = 2
\end{aligned}
$$

Back