The Kirchhoff Index of Cluster Networks

C. Araúz[1], E. Bendito[1], A. Carmona[1] and A.M. Encinas[1]

Budapest, Eurocomb 2011

[1]Dept. Matemàtica Aplicada III
Universitat Politècnica de Catalunya, Barcelona

Kirchhoff Index in Chemistry

⇝ The Kirchhoff Index was introduced in Chemistry

Kirchhoff Index in Chemistry

- The Kirchhoff Index was introduced in Chemistry
- Kirchhoff Index of a molecule:

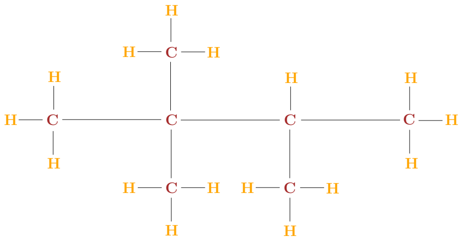
$$\sum_{\text{atoms}} \left(\begin{array}{c} \text{atomic displacements} \\ \text{from equilibrium positions} \end{array} \right)^2$$

Kirchhoff Index in Chemistry

- The Kirchhoff Index was introduced in Chemistry
- Kirchhoff Index of a molecule:

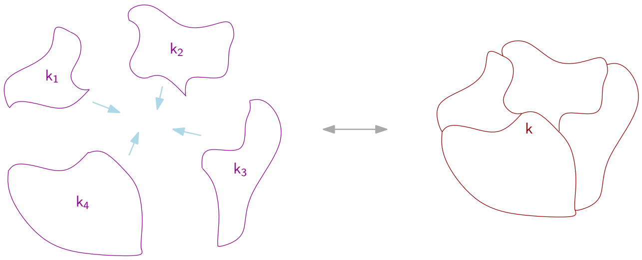
$$\sum_{\text{atoms}} \left(\begin{array}{c} \text{atomic displacements} \\ \text{from equilibrium positions} \end{array} \right)^2$$

small Kirchhoff Index $\Rightarrow$ the atoms are very rigid in the molecule



Kirchhoff Index in Mathematics

⇝ In the mathematic field, it is interesting to find possible relations between the Kirchhoff Indexes of composite networks and those of their factors



Our objective

⇝ We have worked with a generalization of the classical Kirchhoff Index

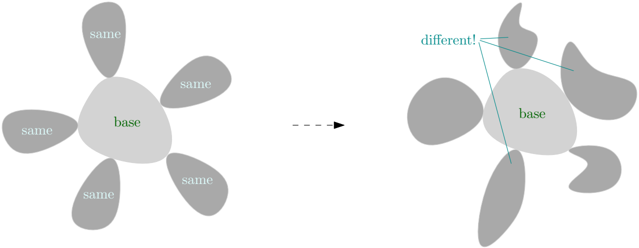
$$\mathsf{k} \quad \dashrightarrow \quad \mathsf{k}(\omega)$$

Our objective

⇝ We have worked with a generalization of the classical Kirchhoff Index

$$\mathsf{k} \quad \dashrightarrow \quad \mathsf{k}(\omega)$$

⇝ We have worked with on a particular family of composite networks: a generalized notion of cluster networks

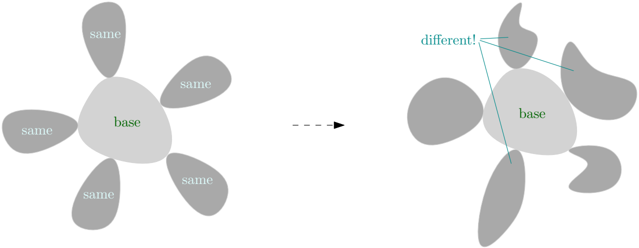


Our objective

⇝ We have worked with a generalization of the classical Kirchhoff Index

$$\mathsf{k} \quad \dashrightarrow \quad \mathsf{k}(\omega)$$

⇝ We have worked with on a particular family of composite networks: a generalized notion of cluster networks



⇝ Our objective is to determine the Kirchhoff Index of generalized cluster networks in terms of the ones of their factors.

Notations and basic results

- $\Gamma = (V, E, c)$ finite connected network
  - $V = \{x_1, \ldots, x_n\}$
  - $\{x_i, x_j\} \in E$ has conductance $\mathsf{c}_{ij} > 0$

Notations and basic results

- $\Gamma = (V, E, c)$ finite connected network
  - $V = \{x_1, \ldots, x_n\}$
  - $\{x_i, x_j\} \in E$ has conductance $\mathsf{c_{ij}} > 0$

- $\omega \in \mathbb{R}^n$, $\quad \omega_i > 0$, $\quad \sum_{i=1}^{n} \omega_i^2 = 1 \quad$ weight on V

Notations and basic results

- $\Gamma = (V, E, c)$ finite connected network
  - $V = \{x_1, \dots, x_n\}$
  - $\{x_i, x_j\} \in E$ has conductance $\mathsf{c_{ij}} > 0$

- $\omega \in \mathbb{R}^n$, $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i^2 = 1$ weight on V

- $\langle \cdot, \cdot \rangle$ standard inner product on $\mathbb{R}^n$

  where $\langle \mathsf{u}, \mathsf{v} \rangle = \sum_{\mathsf{i}=1}^{n} \mathsf{u_i v_i}$ for $\mathsf{u}, \mathsf{v} \in \mathbb{R}^n$

Notations and basic results

- $\Gamma = (V, E, c)$ finite connected network
  - $V = \{x_1, \dots, x_n\}$
  - $\{x_i, x_j\} \in E$ has conductance $\mathsf{c_{ij}} > 0$

- $\omega \in \mathbb{R}^n$, $\quad \omega_i > 0$, $\quad \sum_{i=1}^{n} \omega_i^2 = 1 \quad$ weight on V

- $\langle \cdot, \cdot \rangle \quad$ standard inner product on $\mathbb{R}^n$

  where $\quad \langle \mathsf{u}, \mathsf{v} \rangle = \sum_{\mathsf{i}=1}^{n} \mathsf{u_i v_i} \quad$ for $\quad \mathsf{u}, \mathsf{v} \in \mathbb{R}^n$

- $\mathsf{L} = (\mathsf{L_{ij}})_{\mathsf{i,j}=1}^{n} \quad$ combinatorial Laplacian

  where $\quad \mathsf{L_{ii}} = \sum_{\mathsf{k}=1}^{n} \mathsf{c_{ik}} \quad$ and $\quad \mathsf{L_{ij}} = -\mathsf{c_{ij}}$ for $\mathsf{i} \neq \mathsf{j}$

Schrödinger matrix

⋇ $\mathsf{L_q} = \mathsf{L} + \mathsf{Q}$ Schrödinger matrix with potential q

- $\mathsf{q} \in \mathbb{R}^n$ potential on Γ
- $\mathsf{Q} = \mathrm{diag}(\mathsf{q}_1, \ldots, \mathsf{q}_n)$ diagonal matrix given by the values of q on V

Schrödinger matrix

- $\mathsf{L_q} = \mathsf{L} + \mathsf{Q}$ Schrödinger matrix with potential q
  - $\mathsf{q} \in \mathbb{R}^n$ potential on Γ
  - $\mathsf{Q} = \mathrm{diag}(\mathsf{q}_1, \ldots, \mathsf{q}_n)$ diagonal matrix given by the values of q on V

Remark

But we want $\mathsf{L_q}$ to be positive semi-definite!

Schrödinger matrix

- $\mathsf{L_q} = \mathsf{L} + \mathsf{Q}$ Schrödinger matrix with potential q
  - $\mathsf{q} \in \mathbb{R}^n$ potential on Γ
  - $\mathsf{Q} = \operatorname{diag}(\mathsf{q}_1, \ldots, \mathsf{q_n})$ diagonal matrix given by the values of q on V

Remark

But we want $\mathsf{L_q}$ to be positive semi-definite!

⇝ We take $\mathsf{q} = \mathsf{q}_\omega \in \mathbb{R}^n$ potential determined by ω, where

$$(\mathsf{q}_\omega)_i = -(\omega)_i^{-1} (\mathsf{L} \cdot \omega)_i$$

Schrödinger matrix

- $\mathsf{L_q} = \mathsf{L} + \mathsf{Q}$ Schrödinger matrix with potential q
  - $\mathsf{q} \in \mathbb{R}^n$ potential on Γ
  - $\mathsf{Q} = \operatorname{diag}(\mathsf{q}_1, \dots, \mathsf{q_n})$ diagonal matrix given by the values of q on V

Remark

But we want $\mathsf{L_q}$ to be positive semi-definite!

⇝ We take $\mathsf{q} = \mathsf{q}_\omega \in \mathbb{R}^n$ potential determined by ω, where

$$(\mathsf{q}_\omega)_i = -(\omega)_i^{-1} (\mathsf{L} \cdot \omega)_i$$

Results

✓ $\mathsf{L}_{\mathsf{q}_\omega}$ *is positive semi-definite and singular*

Schrödinger matrix

- $\mathsf{L_q} = \mathsf{L} + \mathsf{Q}$ Schrödinger matrix with potential q
  - $\mathsf{q} \in \mathbb{R}^n$ potential on Γ
  - $\mathsf{Q} = \mathrm{diag}(\mathsf{q}_1, \dots, \mathsf{q_n})$ diagonal matrix given by the values of q on V

Remark

But we want $\mathsf{L_q}$ *to be positive semi-definite!*

⇝ We take $\mathsf{q} = \mathsf{q}_\omega \in \mathbb{R}^n$ potential determined by ω, where

$$(\mathsf{q}_\omega)_i = -(\omega)_i^{-1}\,(\mathsf{L}\cdot\omega)_i$$

Results

✓ $\mathsf{L}_{\mathsf{q}_\omega}$ *is positive semi-definite and singular*

✓ $\mathsf{L}_{\mathsf{q}_\omega}\cdot \mathsf{v} = 0$ *iff* $\mathsf{v} = a\omega$ *with* $a \in \mathbb{R}$

Green matrix

- ⋇ $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \mathsf{g}$ Poisson equation

Green matrix

※ $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \mathsf{g}$ Poisson equation

※ $\mathsf{L}^\dagger_{\mathsf{q}_\omega}$ Green matrix $\mathsf{f} \in \mathbb{R}^n \xrightarrow{\mathsf{L}^\dagger_{\mathsf{q}_\omega}}$ the unique $\mathsf{u} \in \mathbb{R}^n$ such that

- $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \mathsf{f} - \langle \mathsf{f}, \omega \rangle \cdot \omega$
- $\langle \mathsf{u}, \omega \rangle = 0$

Remark

The Green matrix $\mathsf{L}^\dagger_{\mathsf{q}_\omega}$ *is the Moore-Penrose inverse of the Schrödinger matrix* $\mathsf{L}_{\mathsf{q}_\omega}$.

Green matrix

※ $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \mathsf{g}$ Poisson equation

※ $\mathsf{L}^{\dagger}_{\mathsf{q}_\omega}$ Green matrix $\mathsf{f} \in \mathbb{R}^n \xrightarrow{\mathsf{L}^{\dagger}_{\mathsf{q}_\omega}}$ the unique $\mathsf{u} \in \mathbb{R}^n$ such that

- $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \mathsf{f} - \langle \mathsf{f}, \omega \rangle \cdot \omega$
- $\langle \mathsf{u}, \omega \rangle = 0$

Remark

The Green matrix $\mathsf{L}^{\dagger}_{\mathsf{q}_\omega}$ *is the Moore-Penrose inverse of the Schrödinger matrix* $\mathsf{L}_{\mathsf{q}_\omega}$.

Result

✓ $\mathsf{L}^{\dagger}_{\mathsf{q}_\omega} \cdot \omega = 0$

Effective resistances and Kirchhoff Index

⁕ R_ω effective resistance between a pair of vertices

$$R_\omega(x_i, x_j) = \mathsf{u}^\top \cdot \mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \frac{\mathsf{u_i}}{\omega_i} - \frac{\mathsf{u_j}}{\omega_j}$$

where $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \dfrac{\mathsf{e_i}}{\omega_i} - \dfrac{\mathsf{e_j}}{\omega_j}$

Effective resistances and Kirchhoff Index

※ R_ω effective resistance between a pair of vertices

$$R_\omega(x_i, x_j) = \mathsf{u}^\top \cdot \mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \frac{\mathsf{u_i}}{\omega_i} - \frac{\mathsf{u_j}}{\omega_j}$$

where $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \dfrac{\mathsf{e_i}}{\omega_i} - \dfrac{\mathsf{e_j}}{\omega_j}$

※ r_ω total resistance on a vertex

$$r_\omega(x_i) = \mathsf{v}^\top \cdot \mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{v} = \frac{\mathsf{v_i}}{\omega_i} - \frac{1}{n}\langle \mathsf{v}, \omega \rangle$$

where $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{v} = \dfrac{\mathsf{e_i}}{\omega_i} - \omega$

Effective resistances and Kirchhoff Index

- R_ω effective resistance between a pair of vertices

$$R_\omega(x_i, x_j) = \mathsf{u}^\top \cdot \mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \frac{\mathsf{u_i}}{\omega_i} - \frac{\mathsf{u_j}}{\omega_j}$$

where $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{u} = \frac{\mathsf{e_i}}{\omega_i} - \frac{\mathsf{e_j}}{\omega_j}$

- r_ω total resistance on a vertex

$$r_\omega(x_i) = \mathsf{v}^\top \cdot \mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{v} = \frac{\mathsf{v_i}}{\omega_i} - \frac{1}{n}\langle \mathsf{v}, \omega \rangle$$

where $\mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{v} = \frac{\mathsf{e_i}}{\omega_i} - \omega$

- $\mathsf{k}(\omega)$ Kirchhoff index of Γ respect to ω

$$\mathsf{k}(\omega) = \frac{1}{2} \sum_{i,j=1}^{n} R_\omega(x_i, x_j)\omega_i^2 \omega_j^2$$

Effective resistances and Kirchhoff Index

Results

✓ $r_\omega > 0$

Effective resistances and Kirchhoff Index

Results

✓ $r_\omega > 0$

✓ R_ω *is a distance on* Γ

Effective resistances and Kirchhoff Index

Results

✓ $r_\omega > 0$

✓ R_ω *is a distance on* Γ

⇝ *We can express them by means of the Green matrix entries*

Effective resistances and Kirchhoff Index

Results

- ✓ $r_\omega > 0$
- ✓ R_ω *is a distance on* Γ

⇝ *We can express them by means of the Green matrix entries*

- ✓ $r_\omega(x_i) = \dfrac{(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ii}}}{\omega_{\mathsf{i}}^2}$

Effective resistances and Kirchhoff Index

Results

- ✓ $r_\omega > 0$
- ✓ R_ω *is a distance on* Γ

⇝ *We can express them by means of the Green matrix entries*

- ✓ $r_\omega(x_i) = \dfrac{(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ii}}}{\omega_{\mathsf{i}}^2}$
- ✓ $R_\omega(x_i, x_j) = r_\omega(x_i) + r_\omega(x_j) + \dfrac{2(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ij}}}{\omega_{\mathsf{i}}\omega_{\mathsf{j}}}$

Effective resistances and Kirchhoff Index

Results

- ✓ $r_\omega > 0$
- ✓ R_ω *is a distance on* Γ

⇝ *We can express them by means of the Green matrix entries*

- ✓ $r_\omega(x_i) = \dfrac{(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ii}}}{\omega_{\mathsf{i}}^2}$
- ✓ $R_\omega(x_i, x_j) = r_\omega(x_i) + r_\omega(x_j) + \dfrac{2(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ij}}}{\omega_{\mathsf{i}}\omega_{\mathsf{j}}}$

⇝ *That allows us to express the Kirchhoff index in terms of the Green matrix!*

Effective resistances and Kirchhoff Index

Results

✓ $r_\omega > 0$

✓ R_ω *is a distance on* Γ

⇝ *We can express them by means of the Green matrix entries*

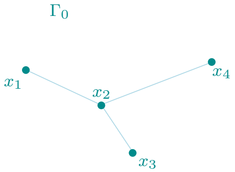
✓ $r_\omega(x_i) = \dfrac{(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ii}}}{\omega_{\mathsf{i}}^2}$

✓ $R_\omega(x_i, x_j) = r_\omega(x_i) + r_\omega(x_j) + \dfrac{2(\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ij}}}{\omega_{\mathsf{i}}\omega_{\mathsf{j}}}$

⇝ *That allows us to express the Kirchhoff index in terms of the Green matrix!*

✓ $\mathsf{k}(\omega) = \sum_{\mathsf{i}=1}^{n} (\mathsf{L}^\dagger_{\mathsf{q}_\omega})_{\mathsf{ii}} = tr(\mathsf{L}^\dagger_{\mathsf{q}_\omega})$

Cluster networks

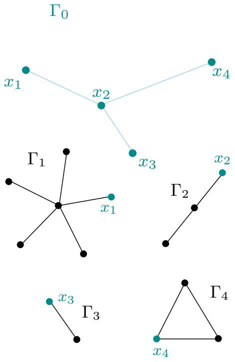


- Γ_0 base network, with $V_0 = \{x_1, \ldots, x_m\}$

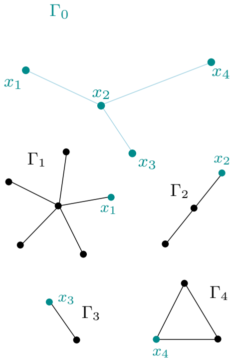
Cluster networks

⁜ Γ_0 base network, with $V_0 = \{x_1, \ldots, x_m\}$

⁜ $\Gamma_1, \ldots, \Gamma_m$ satellite networks, with $x_i \in V_i$



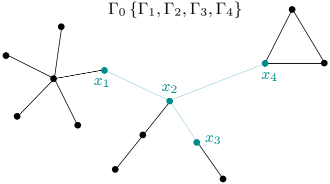
Cluster networks



- Γ_0 base network, with $V_0 = \{x_1, \dots, x_m\}$

- $\Gamma_1, \dots, \Gamma_m$ satellite networks, with $x_i \in V_i$

- $\Gamma = (V, E, c) = \Gamma_0\{\Gamma_1, \dots, \Gamma_m\}$ Cluster network
  - $V = \bigcup\limits_{i=1}^{m} V_i$
  - $E = \bigcup\limits_{i=0}^{m} E_i$
  - c given by the original conductances



Cluster networks

* ω weight on Γ

Cluster networks

⋇ ω weight on Γ

σ_i coefficients such that $\omega(i)$ weight on Γ_i

$$\omega(i)_j = \sigma_i^{-1} \omega_j$$

Cluster networks

- ω weight on Γ

  σ_i coefficients such that $\omega(i)$ weight on Γ_i

  $$\omega(i)_j = \sigma_i^{-1}\omega_j$$

- L combinatorial laplacian of Γ, L^{i} combinatorial laplacian of Γ_i

Cluster networks

- ω weight on Γ

  σ_i coefficients such that $\omega(i)$ weight on Γ_i

  $$\omega(i)_j = \sigma_i^{-1}\omega_j$$

- L combinatorial laplacian of Γ, L^{i} combinatorial laplacian of Γ_i

- q_ω potential on Γ, $(\mathsf{q}_\omega)_{\mathsf{i}} = -\omega_{\mathsf{i}}^{-1}(\mathsf{L}\cdot\omega)_{\mathsf{i}}$

  $\mathsf{q}_{\omega(\mathsf{i})}$ potential on Γ_i, $(\mathsf{q}_\omega(\mathsf{i}))_{\mathsf{j}} = -\omega(\mathsf{i})_{\mathsf{j}}^{-1}\left(\mathsf{L}^{\mathsf{i}}\cdot\omega(\mathsf{i})\right)_{\mathsf{j}}$

Cluster networks

⋇ ω weight on Γ

σ_i coefficients such that $\omega(i)$ weight on Γ_i

$$\omega(i)_j = \sigma_i^{-1}\omega_j$$

⋇ L combinatorial laplacian of Γ, L^{i} combinatorial laplacian of Γ_i

⋇ q_ω potential on Γ, $(\mathsf{q}_\omega)_{\mathsf{i}} = -\omega_{\mathsf{i}}^{-1}(\mathsf{L}\cdot\omega)_{\mathsf{i}}$

$\mathsf{q}_{\omega(\mathsf{i})}$ potential on Γ_i, $(\mathsf{q}_\omega(\mathsf{i}))_{\mathsf{j}} = -\omega(\mathsf{i})_{\mathsf{j}}^{-1}\left(\mathsf{L}^{\mathsf{i}}\cdot\omega(\mathsf{i})\right)_{\mathsf{j}}$

Results

✓ $\mathsf{L}\cdot\mathsf{u} = \mathsf{L}^{\mathsf{i}}\cdot\mathsf{u} + \left(\mathsf{L}^0\cdot\mathsf{u}\right)_{\mathsf{i}}\cdot\mathsf{e}_{\mathsf{i}}$ *on* Γ_i *for all* $\mathsf{u}\in\mathbb{R}^n$, $\mathsf{i}=1,\dots,m$

Cluster networks

※ ω weight on Γ

σ_i coefficients such that $\omega(i)$ weight on Γ_i

$$\omega(i)_j = \sigma_i^{-1}\omega_j$$

※ L combinatorial laplacian of Γ, L^{i} combinatorial laplacian of Γ_i

※ q_ω potential on Γ, $(\mathsf{q}_\omega)_{\mathsf{i}} = -\omega_{\mathsf{i}}^{-1}(\mathsf{L}\cdot\omega)_{\mathsf{i}}$

$\mathsf{q}_{\omega(\mathsf{i})}$ potential on Γ_i, $(\mathsf{q}_\omega(\mathsf{i}))_{\mathsf{j}} = -\omega(\mathsf{i})_{\mathsf{j}}^{-1}\left(\mathsf{L}^{\mathsf{i}}\cdot\omega(\mathsf{i})\right)_{\mathsf{j}}$

Results

✓ $\mathsf{L}\cdot\mathsf{u} = \mathsf{L}^{\mathsf{i}}\cdot\mathsf{u} + \left(\mathsf{L}^0\cdot\mathsf{u}\right)_{\mathsf{i}}\cdot\mathsf{e}_{\mathsf{i}}$ *on* Γ_i *for all* $\mathsf{u}\in\mathbb{R}^n$, $\mathsf{i} = 1,\dots,m$

✓ $(\mathsf{q}_\omega)_{\mathsf{j}} = (\mathsf{q}_{\omega(\mathsf{i})})_{\mathsf{j}} + (\mathsf{q}_{\omega(0)})_{\mathsf{i}}\cdot\mathsf{e}_{\mathsf{i}}$ *on* Γ_i

Cluster networks

⁕ ω weight on Γ

σ_i coefficients such that $\omega(i)$ weight on Γ_i

$$\omega(i)_j = \sigma_i^{-1}\omega_j$$

⁕ L combinatorial laplacian of Γ, L^{i} combinatorial laplacian of Γ_i

⁕ q_ω potential on Γ, $(\mathsf{q}_\omega)_{\mathsf{i}} = -\omega_{\mathsf{i}}^{-1}(\mathsf{L}\cdot\omega)_{\mathsf{i}}$

$\mathsf{q}_{\omega(\mathsf{i})}$ potential on Γ_i, $(\mathsf{q}_\omega(\mathsf{i}))_{\mathsf{j}} = -\omega(\mathsf{i})_{\mathsf{j}}^{-1}\left(\mathsf{L}^{\mathsf{i}}\cdot\omega(\mathsf{i})\right)_{\mathsf{j}}$

Results

✓ $\mathsf{L}\cdot\mathsf{u} = \mathsf{L}^{\mathsf{i}}\cdot\mathsf{u} + \left(\mathsf{L}^0\cdot\mathsf{u}\right)_{\mathsf{i}}\cdot\mathsf{e}_{\mathsf{i}}$ *on* Γ_i *for all* $\mathsf{u}\in\mathbb{R}^n$, $\mathsf{i}=1,\dots,m$

✓ $(\mathsf{q}_\omega)_{\mathsf{j}} = (\mathsf{q}_{\omega(\mathsf{i})})_{\mathsf{j}} + (\mathsf{q}_{\omega(0)})_{\mathsf{i}}\cdot\mathsf{e}_{\mathsf{i}}$ *on* Γ_i

✓ *Therefore,* $\mathsf{L}_{\mathsf{q}_\omega}\cdot\mathsf{u} = \mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}}\cdot\mathsf{u} + \left(\mathsf{L}^0{}_{\mathsf{q}_{\omega(\mathsf{i})}}\cdot u\right)_i\cdot e_i$ *on* Γ_i

Cluster's Green matrix and Kirchhoff Index

Result

Given $\mathsf{f} \in \mathbb{R}^n$ *with* $\langle \omega, \mathsf{f} \rangle = 0$

Cluster's Green matrix and Kirchhoff Index

Result

Given $\mathsf{f} \in \mathbb{R}^n$ *with* $\langle \omega, \mathsf{f} \rangle = 0$, *consider* $\mathsf{L}_{\mathsf{q}_\omega} \cdot u = f$ *on* Γ *(Poisson equation)*

Cluster's Green matrix and Kirchhoff Index

Result

Given $\mathsf{f} \in \mathbb{R}^n$ *with* $\langle \omega, \mathsf{f} \rangle = 0$, *consider* $\mathsf{L}_{\mathsf{q}_\omega} \cdot u = f$ *on* Γ *(Poisson equation)*

Then,

$$\mathsf{u} = \sum_{\mathsf{i}=1}^{m} \left((\mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}})^{\dagger} \cdot \mathsf{f} - A_{\mathsf{i}} (\mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}})^{\dagger} \cdot \mathsf{e}_{\mathsf{i}} \right) + \sum_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{B}_{\mathsf{i}} \left(\left((\mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}})^{\dagger} \cdot \mathsf{f} \right)_{\mathsf{i}} - \mathsf{A}_{\mathsf{i}} (\mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}})^{\dagger}_{\mathsf{ii}} - \mathsf{C}_{\mathsf{i}} \right) [\sigma_{\mathsf{i}} \omega - \omega(\mathsf{i})]$$

is the unique solution with $\langle \omega, \mathsf{f} \rangle = 0$.

Remark

We now know how to solve Poisson equations on a cluster network only by knowing the Green matrices of the factors

Cluster's Green matrix and Kirchhoff Index

⇝ Green matrix of the cluster network in terms of the ones of the factors

Result

$$\left(\mathsf{L}^{\dagger}_{\mathsf{q}_\omega}\right)_{\mathsf{kl}} = \left(\mathsf{L}^{\mathsf{j}}_{\mathsf{q}_{\omega(\mathsf{j})}}\right)^{\dagger}_{\mathsf{kl}} + \mathsf{D}_{\mathsf{ij}}\left(\mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}}\right)^{\dagger}_{\mathsf{ki}} + \mathsf{D}_{\mathsf{jl}}\left(\mathsf{L}^{\mathsf{j}}_{\mathsf{q}_{\omega(\mathsf{j})}}\right)^{\dagger}_{\mathsf{lj}} - \mathsf{F}_{\mathsf{jk}}(\mathsf{L}^{\mathsf{j}}_{\mathsf{q}_{\omega(\mathsf{j})}})^{\dagger}_{\mathsf{kj}} - \mathsf{F}_{\mathsf{il}}(\mathsf{L}^{\mathsf{i}}_{\mathsf{q}_{\omega(\mathsf{i})}})^{\dagger}_{\mathsf{li}} + \mathsf{g}_{\mathsf{ji}}\omega(\mathsf{i})\cdot\omega(\mathsf{j})^{\top}$$

is the Green matrix on $V_i \times V_j$

Cluster's Green matrix and Kirchhoff Index

- Green matrix of the cluster network in terms of the ones of the factors

Result

$$\left(L^{\dagger}_{q_\omega}\right)_{kl} = \left(L^{j}_{q_{\omega(j)}}\right)^{\dagger}_{kl} + D_{ij}\left(L^{i}_{q_{\omega(i)}}\right)^{\dagger}_{ki} + D_{jl}\left(L^{j}_{q_{\omega(j)}}\right)^{\dagger}_{lj} - F_{jk}(L^{j}_{q_{\omega(j)}})^{\dagger}_{kj} - F_{il}(L^{i}_{q_{\omega(i)}})^{\dagger}_{li} + g_{ji}\omega(i)\cdot\omega(j)^{\top}$$

is the Green matrix on $V_i \times V_j$

- Kirchhoff Index of the cluster network in terms of the factors

Result

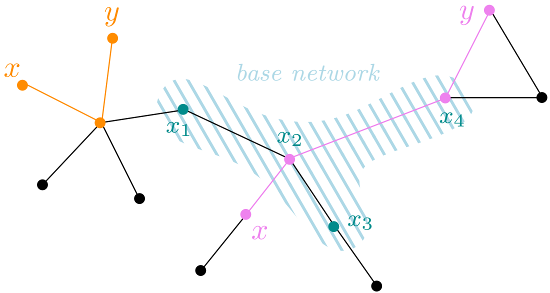
$$k(\omega) = \sum_{i=1}^{m} k_i(\omega(i)) + \frac{1}{2\sigma_0^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_i^2\sigma_j^2 R_{\omega(0)}(x_i,x_j) + \sum_{i=1}^{m}(1-\sigma_i^2)r_{\omega(i)}(x_i)$$

Cluster's effective resistances

Result

$$R_\omega(x,y) = \frac{R_{\omega(i)}(x,y)}{\sigma_i^2} \text{ if } x,y \in V_i,$$

$$R_\omega(x,y) = \frac{R_{\omega(i)}(x,x_i)}{\sigma_i^2} + \frac{R_{\omega(0)}(x_i,x_j)}{\sigma_0^2} + \frac{R_{\omega(j)}(x_j,y)}{\sigma_j^2} \text{ if } x \in V_i, y \in V_j, i \neq j$$



Applications

⤳ Restricting this cluster network to a classical cluster graph, with same distinguished vertex on every satellite

Applications

$\rightsquigarrow$ Restricting this cluster network to a classical cluster graph, with same distinguished vertex on every satellite

$$\Downarrow$$

we get the formula obtained by Li, Yang and Zhang
for the classical cluster Kirchhoff Index (weight-adapted)

$$k(\omega) = \quad k_1(\omega(1)) + nk_0(\omega(0)) + \frac{m-1}{n} \sum_{x \in V_1} R_{\omega(1)}(x, x_1).$$

Thanks for your attention!