The M–matrix inverse problem for singular and symmetric Jacobi matrices

E. Bendito[1], A. Carmona[1], A.M. Encinas[1], and M. Mitjana[2]

[1] *Departament de Matemàtica Aplicada III*
[2] *Departament de Matemàtica Aplicada I*
Universitat Politècnica de Catalunya.
Mod. C2, Campus Nord
C/ Jordi Girona Salgado 1–3
08034 Barcelona. Spain
Phone: +34 93 401 69 13
e-mail: enrique.bendito@upc.edu, angeles.carmona@upc.edu,
andres.marcos.encinas@upc.edu, margarida.mitjana@upc.edu
Corresponding author: Margarida Mitjana

Abstract

We aim here at characterizing when the Moore–Penrose inverse of a singular and symmetric Jacobi M–matrix is also an M–matrix. This characterization involves a highly non–linear system of inequalities on the off–diagonal entries of the matrix. We obtain all the solutions of this system for $n \leq 3$ but when $n \geq 4$, the system becomes much more complicated. Our main result establishes that for any n, there exist singular, symmetric and tridiagonal M–matrices of order n whose Moore–Penrose inverse is also an M–matrix.

Keywords: M–matrix, Moore–Penrose inverse, Laplacian, tridiagonal–matrix.

1 Statement of the Problem

The matrices that can be expressed as $\mathsf{M} = k\mathsf{I} - \mathsf{A}$, where $k > 0$ and $\mathsf{A} \geq 0$, appear in relation with systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics.

In the graph theory framework the most studied cases are the combinatorial Laplacian of a k–regular graph, where A is its adjacency matrix and the probabilistic Laplacian, where $k = 1$ and A is the transition matrix for a Markov chain whose states are the vertices of the graph. If k is at least the spectral radio of A, then M is called an *M–matrix*. This type of matrices arise naturally in some discretizations of differential operators, particularly those with a minimum/maximum principle, such as the Laplacian, and as such are well–studied in scientific computing. In fact, M–matrices satisfy monotonicity properties that are the discrete counterpart of the minimum principle, and this makes them suitable for the resolution of large sparse systems of linear equations by iterative methods.

A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative, [3]. However, when the matrix is an irreducible and singular M–matrix this property does not hold for any generalized inverse. For instance, the Moore–Penrose inverse of the combinatorial Laplacian of a path of length ≥ 4 has non–negative off diagonal entries.

A (finite) Jacobi matrix is a tridiagonal matrix. This type of matrices usually appears in relation with second order linear difference equations and with orthogonal polynomials on the line.

Given $c_1, \dots, c_{n-1} > 0$ and $d_1, \dots, d_n \geq 0$ such that the tridiagonal matrix

$$\mathsf{M} = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-1} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix} \tag{1}$$

is a singular M–matrix, we aim here at determining when its Moore–Penrose inverse $\mathsf{M}^\dagger$ is also an M–matrix. This problem has been widely studied for several families of this type of matrices, see for instance [2, 5, 6, 7]. Although this problem lies in the framework of linear algebra, we have tackled it by applying methods from the operator theory on finite networks, see [2]. To do this we take into account that the off–diagonal entries of M can be identified with the conductance function of a weighted n–path. Specifically, if $V_n = \{x_1, \dots, x_n\}$, then we can consider the weighted path $\Gamma = (V_n, c)$ where the *conductance* between vertices x_i and x_{i+1} is defined by $c(x_i, x_{i+1}) = c_i$.

Each real function on V_n can be identified with a (column) vector of $\mathbb{R}^n$ and hence each endomorphism of the space of real functions on V_n can be identified with a matrix of order n and conversely. In particular, the matrix obtained by choosing $d_1 = c_1$, $d_n = c_{n-1}$ and $d_i = c_{i-1} + c_i$ for $i = 2, \dots, n-1$ is nothing but the combinatorial Laplacian of the network Γ. Therefore, M can be considered as a *perturbed Laplacian of* Γ in the sense of [1] and then we ask which perturbed Laplacians of Γ are singular and positive semi–definite and their Moore–Penrose inverse also is.

From the operator theory point of view, the perturbed Laplacians are identified with the so–called *discrete Schrödinger operators of* Γ, see for instance [4] and references therein for several physical interpretations. In addition, this terminology suggests some sort of relationship with the differential operators with the same name. In fact, many of the techniques and results in this framework appear as the discrete counterpart of the standard treatment of the resolvent of elliptic operators on Riemannian manifolds, see [2].

In this paper we characterize when the Moore–Penrose inverse of a singular, symmetric and tridiagonal M–matrix is also an M–matrix. This characterization involves a highly non–linear system of inequalities on the off–diagonal entries of the matrix. We obtain all the solutions of this system for $n \leq 3$. For $n \geq 4$, the system becomes much more complicated and the key idea to solve it is to apply well–known properties of general M–matrices to the coefficient matrix of the system. Our main result establishes that for any n, there exist singular, symmetric and tridiagonal M–matrices of order n whose Moore–Penrose inverse is also an M–matrix.

2 The M–inverse problem

It was proved in [2] that the matrix given in (1) is a singular M–matrix iff there exists $\omega_1,\dots,\omega_n>0$ such that $\omega_1^2+\ldots+\omega_n^2=1$ and

$$d_1=\frac{c_1\omega_2}{\omega_1},\quad d_n=\frac{c_{n-1}\omega_{n-1}}{\omega_n}\quad\text{and}\quad d_j=\frac{1}{\omega_j}(c_j\omega_{j+1}+c_{j-1}\omega_{j-1}) \tag{2}$$

for any $j=2,\dots,n-1$. Moreover, the weight is uniquely determined by $(d_1,\dots,d_n)$ and $(c_1,\dots,c_{n-1})$.

In the sequel, any $\mathsf{c}=(c_1,\dots,c_{n-1})\in(0,+\infty)^{n-1}$ and any $\omega=(\omega_1,\dots,\omega_n)\in(0,+\infty)^n$ such that $\omega_1^2+\ldots+\omega_n^2=1$ are called *conductance* and *weight*, respectively. The set of weights is denoted by $\Omega(V_n)$. Moreover, the matrix given in (1), where c is a conductance, $\omega\in\Omega(V_n)$ and the diagonal entries are given by (2), is denoted by $\mathsf{M}(\mathsf{c},\omega)$ and hence its Moore–Penrose inverse is denoted by $\mathsf{M}^\dagger(\mathsf{c},\omega)$. Given a conductance c, the set of weights such that $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is an M–matrix is denoted by $\Omega(c)$, whereas given a weight ω, the set of conductances such that $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is an M–matrix is denoted by $\mathsf{C}(\omega)$. Therefore, $\omega\in\Omega(\mathsf{c})$ iff $\mathsf{c}\in\mathsf{C}(\omega)$. We drop ω in all the expressions when ω is constant; that is when $\omega_j=\dfrac{1}{\sqrt{n}}$, for any $j=1,\dots,n$.

Throughout the paper, we use the conventions $\sum\limits_{l=i}^{j}a_l=0$ and $\prod\limits_{l=i}^{j}a_l=1$ when $j<i$. In addition we denote by e_j the j-th vector in the standard basis of $\mathbb{R}^n$ and by e the vector $\mathsf{e}=\mathsf{e}_1+\cdots+\mathsf{e}_n$.

Proposition 2.1 ([2, Corollary 5.2]) *The Moore–Penrose inverse of* $\mathsf{M}(\mathsf{c},\omega)$ *is* $\mathsf{M}^\dagger(\mathsf{c},\omega)=(g_{ij})$, *where*

$$g_{ji}=g_{ij}=\omega_i\omega_j\left[\sum_{k=1}^{i-1}\frac{\left(\sum\limits_{l=1}^{k}\omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}}+\sum_{k=i}^{n-1}\frac{\left(\sum\limits_{l=k+1}^{n}\omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}}-\sum_{k=i}^{j-1}\frac{\left(\sum\limits_{l=k+1}^{n}\omega_l^2\right)}{c_k\omega_k\omega_{k+1}}\right]$$

for any $1\le i\le j\le n$.

The Moore–Penrose inverse for the normalized Laplacian; that is, when ω is the square root of the generalized degree, was obtained in [8, Theorem 9].

If we take into account that the Moore–Penrose inverse of a symmetric and positive semi–definite matrix is itself symmetric and positive semi–definite, as a by–product of the expression of $\mathsf{M}^\dagger(\mathsf{c},\omega)$ we can easily characterize when it is an M–matrix.

Theorem 2.2 $\mathsf{M}^\dagger(\mathsf{c},\omega)$ *is an M–matrix iff* $g_{ii+1}\le 0$ *for any* $i=1,\dots,n-1$, *that is; iff*

$$\frac{\left(\sum\limits_{l=i+1}^{n}\omega_l^2\right)\left(\sum\limits_{l=1}^{i}\omega_l^2\right)}{c_i\omega_i\omega_{i+1}}\ge\sum_{k=1}^{i-1}\frac{\left(\sum\limits_{l=1}^{k}\omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}}+\sum_{k=i+1}^{n-1}\frac{\left(\sum\limits_{l=k+1}^{n}\omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}},\quad i=1,\dots,n-1.$$

The conclusion of the above Theorem for ω constant was given in [5, Lemma 3.1].

The following result is a straightforward consequence of the above result.

Corollary 2.3 *For $n = 2$, $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is always an M–matrix. In fact, for any $c > 0$ and any $0 < x < 1$, if $\omega = \big(x, \sqrt{1-x^2}\big)$, we get*

$$\mathsf{M}(\mathsf{c},\omega) = c\begin{bmatrix} \frac{\sqrt{1-x^2}}{x} & -1 \\ -1 & \frac{x}{\sqrt{1-x^2}} \end{bmatrix} \quad \textit{and} \quad \mathsf{M}^\dagger(\mathsf{c},\omega) = \frac{x\sqrt{1-x^2}}{c}\begin{bmatrix} 1-x^2 & -x\sqrt{1-x^2} \\ -x\sqrt{1-x^2} & x^2 \end{bmatrix}.$$

Corollary 2.4 *When $n = 3$, $\mathsf{M}^\dagger(\mathsf{c},\omega)$ is an M–matrix iff*

$$\frac{\omega_1^3}{\omega_3(1-\omega_3^2)} \le \frac{c_1}{c_2} \le \frac{\omega_1(1-\omega_1^2)}{\omega_3^3}.$$

In particular, if ω is constant, then $\mathsf{M}^\dagger(\mathsf{c})$ is an M–matrix iff $\dfrac{1}{2} \le \dfrac{c_1}{c_2} \le 2$. On the other hand, for any conductance c, it is satisfied that

$$\Omega(\mathsf{c}) = \left\{\Big(\omega_1, \sqrt{1-(1+t^2)\omega_1^2}, t\omega_1\Big) : 0 < t < \frac{c_2}{c_1},\ \ 0 < \omega_1 \le \sqrt{\frac{tc_1}{c_2+t^3c_1}}\right\}$$
$$\bigcup \left\{\Big(\omega_1, \sqrt{1-(1+t^2)\omega_1^2}, t\omega_1\Big) : \frac{c_2}{c_1} \le t,\ \ 0 < \omega_1 < \sqrt{\frac{c_2}{c_2+t^3c_1}}\right\}.$$

Proof. In this case the system of inequalities in Theorem 2.2 becomes

$$\frac{\omega_1^2(1-\omega_1^2)}{c_1\omega_1\omega_2} \ge \frac{\omega_3^4}{c_2\omega_2\omega_3} \quad \text{and} \quad \frac{\omega_3^2(1-\omega_3^2)}{c_2\omega_2\omega_3} \ge \frac{\omega_1^4}{c_1\omega_1\omega_2},$$

that are equivalent to the claimed inequalities.

On the other hand, given $\omega \in \Omega(V_3)$, if we consider $t = \dfrac{\omega_3}{\omega_1}$, necessarily $\omega_2 = \sqrt{1-(1+t^2)\omega_1^2}$ and $0 < \omega_1 < \sqrt{\dfrac{1}{1+t^2}}$. Moreover, given a conductance c and fixed $t > 0$ and $0 < x < \sqrt{\dfrac{1}{1+t^2}}$, then $\Big(x, \sqrt{1-(1+t^2)x^2}, tx\Big) \in \Omega(\mathsf{c})$ iff $(c_2 + t^3c_1)x^2 \le \min\{tc_1, c_2\}$ and the result follows. ∎

Corollary 2.5 *When the weight is constant, then $\mathsf{M}^\dagger(\mathsf{c})$ is an M–matrix iff $n \le 4$ and moreover either $\dfrac{1}{2} \le \dfrac{c_1}{c_2} \le 2$ when $n = 3$ or $c_1 = c_3$ and $c_2 = 2c_1$ when $n = 4$.*

Proof. When ω is constant, the system of inequalities in Theorem 2.2 is equivalent to

$$\frac{i(n-i)}{c_i} \ge \sum_{k=1}^{i-1}\frac{k^2}{c_k} + \sum_{k=i+1}^{n-1}\frac{(n-k)^2}{c_k}, \qquad i = 1,\dots,n-1.$$

Expanding the above inequalities up to $i = 3$, we get that

$$\frac{(n-1)}{c_1} \ge \sum_{k=2}^{n-1}\frac{(n-k)^2}{c_k}, \qquad \frac{2(n-2)}{c_2} \ge \frac{1}{c_1}+\sum_{k=3}^{n-1}\frac{(n-k)^2}{c_k} \quad \text{and} \quad \frac{3(n-3)}{c_3} \ge \frac{1}{c_1}+\frac{4}{c_2}+\sum_{k=4}^{n-1}\frac{(n-k)^2}{c_k},$$

and hence that

$$\frac{(n-1)}{c_1} \geq \sum_{k=2}^{n-1} \frac{(n-k)^2}{c_k}, \qquad \frac{(n-2)}{c_2} \geq \sum_{k=3}^{n-1} \frac{(n-k)^2}{c_k} \quad \text{and} \quad \frac{2(n-3)}{c_3} \geq \frac{n}{c_2} + \sum_{k=4}^{n-1} \frac{(n-k)^2}{c_k},$$

which implies that

$$\frac{(n-3)(5n-4-n^2)}{c_3} \geq 2(n-1) \sum_{k=4}^{n-1} \frac{(n-k)^2}{c_k}.$$

Therefore, $5n - n^2 - 4 = (n-1)(4-n)$ must be non–negative and this occurs iff $n \leq 4$. Moreover, when $n = 4$, the system of inequalities in Theorem 2.2 is

$$\frac{3}{c_1} \geq \frac{4}{c_2} + \frac{1}{c_3}, \qquad \frac{4}{c_2} \geq \frac{1}{c_1} + \frac{1}{c_3} \quad \text{and} \quad \frac{3}{c_3} \geq \frac{1}{c_1} + \frac{4}{c_2},$$

which implies that

$$\frac{1}{c_1} \geq \frac{1}{c_3}, \qquad \frac{2}{c_2} \geq \frac{1}{c_3} \quad \text{and} \quad \frac{3}{c_3} \geq \frac{1}{c_1} + \frac{4}{c_2} \geq \frac{1}{c_1} + \frac{2}{c_3} \geq \frac{3}{c_3}$$

and hence $c_2 = 2c_3$ and $c_1 = c_3$. Conversely, when ω is constant and $c_1 = c_3$ and $c_2 = 2c_1$, then system of inequalities in Theorem 2.2 is satisfied, and hence $\mathsf{M}^\dagger(\mathsf{c})$ is an M–matrix. ∎

The above result was also obtained in [5] by using a different approach.

3 The general case

The cases ω constant and $n = 2, 3$ for arbitrary weights are the only ones that the system of inequalities in Theorem 2.2 can be solved directly. For $n \geq 4$ we will follow a different way that also works for $n = 2, 3$. If for a conductance c, we define $\mathsf{c}^{-1} = (c_1^{-1}, \dots, c_{n-1}^{-1})^T$, then from Theorem 2.2, $\mathsf{M}^\dagger(\mathsf{c}, \omega)$ is an M–matrix iff all the entries of the vector $\mathsf{A}(\omega)\mathsf{c}^{-1}$ are non–negative, where

$$\mathsf{A}(\omega) = \begin{bmatrix} \frac{W_1(1-W_1)}{\omega_1\omega_2} & -\frac{(1-W_2)^2}{\omega_2\omega_3} & -\frac{(1-W_3)^2}{\omega_3\omega_4} & \cdots & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n} \\ -\frac{W_1^2}{\omega_1\omega_2} & \frac{W_2(1-W_2)}{\omega_2\omega_3} & -\frac{(1-W_3)^2}{\omega_3\omega_4} & \cdots & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\frac{W_1^2}{\omega_1\omega_2} & -\frac{W_2^2}{\omega_2\omega_3} & \cdots & \frac{W_{n-2}(1-W_{n-2})}{\omega_{n-2}\omega_{n-1}} & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n} \\ -\frac{W_1^2}{\omega_1\omega_2} & -\frac{W_2^2}{\omega_2\omega_3} & \cdots & -\frac{W_{n-2}^2}{\omega_{n-2}\omega_{n-1}} & \frac{W_{n-1}(1-W_{n-1})}{\omega_{n-1}\omega_n} \end{bmatrix}$$

and $W_j = \sum\limits_{l=1}^{j} \omega_l^2$.

Observe that $\mathsf{A}(\omega)$ is an irreducible $(n-1)$–order Z–matrix. Therefore, by applying [3, Exercise 6.4.14], if $\mathsf{C}(\omega) \neq \emptyset$ for a weight ω, then $\mathsf{A}(\omega)$ is an M–matrix. Conversely when $\mathsf{A}(\omega)$ is a non singular M–matrix then $\mathsf{c} \in \mathsf{C}(\omega)$ iff $\mathsf{c}^{-1} = \mathsf{A}^{-1}(\omega)\mathsf{a}$, where a is non null and all its entries are non–negative, since from [3, Theorem 6.2.7] all the entries of $\mathsf{A}^{-1}(\omega)$ are positive.

Our next aim is to characterize when $\mathsf{A}(\omega)$ is an M–matrix for a given weight $\omega \in \Omega(V_n)$. To do this, given $\omega \in \Omega(V_n)$ we denote by $\mathsf{c}(\omega)$ the *conductance generated by* ω, whose components are given by

$$c_j(\omega) = \frac{(1-W_j)}{\omega_j\omega_{j+1}} \prod_{k=j}^{n-2} \frac{W_k}{(1-W_k)}, \qquad j = 1,\dots,n-1.$$

In addition, for $2 \le i \le n$ we define the function $D_{n,i}\colon \Omega(V_n) \longrightarrow \mathbb{R}$ by

$$D_{n,i}(\omega) = \omega_{i-1}^2 - \sum_{j=1}^{i-3} W_j \prod_{k=j+1}^{i-2} \frac{(1-W_k)}{W_k}.$$

Moreover for any $\omega \in \Omega(V_n)$, we also consider the values $q_i(\omega) = \omega_{i+1}^2 + D_{n,i+1}(\omega) - D_{n,i+2}(\omega)$, for any $i = 1,\dots,n-2$ and $q_{n-1}(\omega) = \omega_n^2 + D_{n,n}(\omega)$.

Next, we show the main properties of functions $D_{n,i}$ and values $q_i(\omega)$ that we use through the paper.

Lemma 3.1 *Given $n \ge 2$ and $\omega \in \Omega(V_n)$, then the following properties hold:*

(i) $D_{n,2}(\omega) = \omega_1^2$, $D_{n,3}(\omega) = \omega_2^2$, *when* $n \ge 3$ *and* $D_{n,4}(\omega) = \dfrac{\omega_2^2\omega_3^2 - \omega_1^2(\omega_4^2 + \cdots + \omega_n^2)}{\omega_1^2 + \omega_2^2}$, *when* $n \ge 4$.

(ii) $D_{n,i}(\omega) = \left[\dfrac{W_{i-1}}{1-W_{i-1}}\right]\Big[D_{n,i+1}(\omega) + (1-W_i)\Big]$, *for any* $i = 2,\dots,n-1$. *In particular, if* $D_{n,n}(\omega) \ge 0$ *then* $D_{n,i}(\omega) > 0$ *for any* $2 \le i \le n-1$.

(iii) Given $0 < y < \omega_n$ *and* $\omega^y \in \Omega(V_{n+1})$ *defined as* $\omega_j^y = \omega_j$, $j = 1,\dots,n-1$, $\omega_n^y = \sqrt{\omega_n^2 - y^2}$ *and* $\omega_{n+1}^y = y$, *then* $D_{n+1,n+1}(\omega^y) = \dfrac{\omega_n^2 D_{n,n}(\omega)}{\omega_1^2 + \cdots + \omega_{n-1}^2} - y^2$.

(iv) $q_1(\omega) = \omega_1^2$, $q_2(\omega) = \dfrac{\omega_2^4 + \omega_1^2(1-\omega_1^2)}{\omega_1^2 + \omega_2^2}$, $\sum\limits_{i=1}^{n-1} q_i(\omega) = 1$ *and moreover* $q_i(\omega) > 0$ *if* $D_{n,n}(\omega) \ge 0$.

Proof. The proofs of the claims in Part (i) are straightforward. Given $i = 2,\dots,n-1$ we get that

$$D_{n,i+1}(\omega) = \omega_i^2 - \left[\frac{1-W_{i-1}}{W_{i-1}}\right]\left[W_{i-2} + \sum_{j=1}^{i-3} W_j^2 \prod_{k=j+1}^{i-2} \frac{1-W_k}{W_k}\right] = \underbrace{\omega_i^2 - (1-W_{i-1})}_{-(1-W_i)} + \left[\frac{1-W_{i-1}}{W_{i-1}}\right] D_{n,i}(\omega),$$

that concludes the first claim of (ii). In addition if $D_{n,i+1}(\omega) \ge 0$, then the above equality implies that $D_{n,i}(\omega) > 0$ and then the last claim can be proved by regressive induction.

On the other hand, $D_{n+1,n}(\omega^y) = D_{n,n}(\omega)$, since $(\omega_n^y)^2 + (\omega_{n+1}^y)^2 = \omega_n^2$ and then applying equality in part (ii) to ω^y, we get that

$$D_{n,n}(\omega) = D_{n+1,n}(\omega^y) = \frac{(\omega_1^y)^2 + \cdots + (\omega_{n-1}^y)^2}{(\omega_n^y)^2 + (\omega_{n+1}^y)^2}\Big[D_{n+1,n+1}(\omega^y) + (\omega_{n+1}^y)^2\Big]$$

and the claim in part (iii) follows.

The two first claims in Part (iv) are straightforward. Moreover,

$$\sum_{i=1}^{n-1} q_i(\omega) = \omega_n^2 + D_{n,n}(\omega) + \sum_{i=1}^{n-2}\omega_{i+1}^2 + \sum_{i=1}^{n-2}\Big[D_{n,i+1}(\omega) - D_{n,i+2}(\omega)\Big] = \sum_{i=2}^{n}\omega_i^2 + D_{n,2}(\omega) = \sum_{i=1}^{n}\omega_i^2 = 1,$$

where we have taken into account that $D_{n,2}(\omega) = \omega_1^2$.

Finally, when $D_{n,n}(\omega) \geq 0$, then $q_{n-1}(\omega) = \omega_n^2 + D_{n,n}(\omega) > 0$, and moreover for any $i = 1, \ldots, n-2$ we get that $q_i(\omega) = \omega_{i+1}^2 + D_{n,i+1}(\omega) - D_{n,i+2}(\omega) > 0$, since $\omega_{i+1}^2 - D_{n,i+2}(\omega) = \sum_{j=1}^{i-1} W_j \prod_{k=j+1}^{i} \dfrac{(1-W_k)}{W_k} > 0$ and $D_{n,i+1}(\omega) > 0$, from the last claim of Part (ii). ∎

Theorem 3.2 *For any $n \geq 2$ and any weight $\omega \in \Omega(V_n)$, $\mathsf{A}(\omega)\mathsf{c}^{-1}(\omega) = D_{n,n}(\omega)\mathsf{e}$ and $\operatorname{rank}\mathsf{A}(\omega) \geq n-2$. Therefore, $\mathsf{A}(\omega)$ is an M–matrix iff $D_{n,n}(\omega) \geq 0$ and it is singular when the equality holds.*

Proof. To prove the first claims we apply the Gauss Method to matrix $\mathsf{A}(\omega)$. The first step consists in substracting the $(i+1)$–row to the i–row, for any $i = 1, \ldots, n-2$. Secondly, we add to the last row the i–row multiplied by W_i, for any $i = 1, \ldots, n-2$ and the third step consists in dividing each row, except the last one, by its diagonal entry. The last step is to add to the last row the i–row multiplied by $\dfrac{W_i}{\omega_i\omega_{i+1}}\big[\omega_{i+1}^2 - D_{n,i+2}(\omega)\big]$, for any $i = 1, \ldots, n-2$ and then applied Part (ii) of Lemma 3.1. So, if we consider the matrix

$$\mathsf{Q}(\omega) = \begin{bmatrix} \frac{\omega_1\omega_2}{W_1} & -\frac{\omega_1\omega_2}{W_1} & 0 & \cdots & 0 \\ 0 & \frac{\omega_2\omega_3}{W_2} & -\frac{\omega_2\omega_3}{W_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} & -\frac{\omega_{n-2}\omega_{n-1}}{W_{n-2}} \\ q_1(\omega) & q_2(\omega) & \cdots & q_{n-2}(\omega) & q_{n-1}(\omega) \end{bmatrix},$$

then $\mathsf{Q}(\omega)\mathsf{A}(\omega) = \mathsf{B}(\omega)$ where

$$\mathsf{B}(\omega) = \begin{bmatrix} 1 & -\frac{\omega_1(1-W_2)}{\omega_3 W_1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\frac{\omega_2(1-W_3)}{\omega_4 W_2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\frac{\omega_3(1-W_4)}{\omega_5 W_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -\frac{\omega_{n-2}(1-W_{n-1})}{\omega_n W_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\omega_n}{\omega_{n-1}} D_{n,n}(\omega) \end{bmatrix}.$$

Moreover, $\mathsf{Q}(\omega)\mathsf{e} = \mathsf{e}_{n-1}$ and

$$\det \mathsf{Q}(\omega) = \Big(\sum_{j=1}^{n-1} q_j(\omega)\Big)\prod_{k=1}^{n-2}\frac{\omega_k\omega_{k+1}}{W_k} = \frac{\omega_{n-1}}{\omega_1}\Big(\prod_{k=1}^{n-2}\omega_k^2\Big)\Big(\prod_{k=1}^{n-2}W_k\Big)^{-1} > 0.$$

Therefore, $\operatorname{rank} \mathsf{A}(\omega) = \operatorname{rank} \mathsf{B}(\omega) \geq n-2$ and $\det \mathsf{A}(\omega) = \dfrac{\omega_n D_{n,n}(\omega)}{\omega_{n-1} \det \mathsf{Q}(\omega)}$, which implies that $\mathsf{A}(\omega)$ is singular iff $D_{n,n}(\omega) = 0$. In addition,

$$\mathsf{A}(\omega)\mathsf{c}^{-1}(\omega) = \mathsf{Q}^{-1}(\omega)\mathsf{B}(\omega)\mathsf{c}^{-1}(\omega) = D_{n,n}(\omega)\mathsf{Q}^{-1}(\omega)\mathsf{e}_{n-1} = D_{n,n}(\omega)\mathsf{e}$$

and hence, $\mathsf{A}(\omega)$ is an M–matrix when $D_{n,n}(\omega) \geq 0$. Conversely, if $D_{n,n}(\omega) < 0$ then $\mathsf{A}^{-1}(\omega)\mathsf{e} = D_{n,n}^{-1}(\omega)\mathsf{c}^{-1}(\omega)$ and hence, A^{-1} is not an M–matrix. ∎

Motivated for the above Theorem, we split $\Omega(V_n)$, the set of weights on V_n, into the subsets

$$\begin{aligned} \Omega_-(V_n) &= \big\{\omega \in \Omega(V_n) : D_{n,n}(\omega) < 0\big\}, \\ \Omega_0(V_n) &= \big\{\omega \in \Omega(V_n) : D_{n,n}(\omega) = 0\big\}, \\ \Omega_+(V_n) &= \big\{\omega \in \Omega(V_n) : D_{n,n}(\omega) > 0\big\}. \end{aligned}$$

Therefore, $\mathsf{C}(\omega) = \emptyset$ iff $\omega \in \Omega_-(V_n)$, $\mathsf{A}(\omega)$ is a singular M–matrix iff $\omega \in \Omega_0(V_n)$ and $\mathsf{A}(\omega)$ is an invertible M–matrix iff $\omega \in \Omega_+(V_n)$. Moreover, from Lemma 3.1 (i), $\Omega(V_n) = \Omega_+(V_n)$ for $n = 2, 3$, whereas for $n = 4$ we get that $\Omega_0(V_n) = \{\omega \in \Omega(V_n) : \omega_1\omega_4 = \omega_2\omega_3\}$ and $\Omega_+(V_n) = \{\omega \in \Omega(V_n) : \omega_1\omega_4 < \omega_2\omega_3\}$. On the other hand, for any $n \geq 4$ we get that

$$D_{n,n}\Big(\frac{1}{\sqrt{n}}\mathsf{e}\Big) = \frac{1}{n}\left[1 - (n-1)\sum_{j=1}^{n-3} j \binom{n-1}{j}^{-1}\right]$$

and hence when ω is constant, A is an M–matrix iff $n \leq 4$, since $1 - (n-1)(n-3)\binom{n-1}{n-3}^{-1} = \dfrac{4-n}{n-2}$. Therefore, for any $n > 5$, $\Omega_-(V_n)$ contains the constant weight which explains why $\mathsf{C} = \emptyset$ when $n > 5$, as was proved in Corollary 2.5.

So, to prove that for any $n \geq 4$ there exist a conductance c and a weight ω such that $\mathsf{M}^\dagger(\mathsf{c}, \omega)$ is an M–matrix it suffices to prove that one of the sets $\Omega_0(V_n)$ or $\Omega_+(V_n)$ is not empty. More explicity, we have the following result.

Proposition 3.3 *For any $n \geq 4$, the sets $\Omega_0(V_n)$ and $\Omega_+(V_n)$ are non empty. Moreover, when $\omega \in \Omega_0(V_n)$ then $\mathsf{C}(\omega) = \big\{t\mathsf{c}(\omega) : t > 0\big\}$, whereas when $\omega \in \Omega_+(V_n)$ then $\big\{t\mathsf{c}(\omega) : t > 0\big\} \subset \mathsf{C}(\omega)$.*

Proof. We proceed by induction on $n \geq 4$.

The claims are true when $n = 4$, since in this case we know that $D_{4,4}(\omega) = 0$ iff $\omega_1\omega_4 = \omega_2\omega_3$ and that $D_{4,4}(\omega) > 0$ iff $\omega_1\omega_4 < \omega_2\omega_3$.

Suppose now that the result is true for $n \geq 4$ and let $\omega \in \Omega(V_n)$ satisfying that $D_{n,n}(\omega) > 0$. If we take $0 < y \leq \omega_n\sqrt{\dfrac{D_{n,n}(\omega)}{W_{n-1}}}$, then $y < \omega_n$ since for $n \geq 4$, $D_{n,n}(\omega) < \omega_{n-1}^2 < W_{n-1}$. Therefore, if we consider $\omega^y \in \Omega(V_{n+1})$, then Lemma 3.1 (iii) assures that

$$D_{n+1,n+1}(\omega) = \frac{\omega_n^2 D_{n,n}(\omega)}{W_{n-1}} - y^2 \geq 0$$

and the equality holds iff we choose $y = \omega_n \sqrt{\dfrac{D_{n,n}(\omega)}{W_{n-1}}}$.

On the other hand, Theorem 3.2 assures that $\big\{t\mathsf{c}(\omega) : t > 0\big\} \subset \mathsf{C}(\omega)$ when either $\omega \in \Omega_0(V_n)$ or $\omega \in \Omega_+(V_n)$. Moreover, if $\omega \in \Omega_0(V_n)$, then Theorem 3.2 also implies that $\mathsf{A}(\omega)$ is a singular M–matrix whose rank equals $n-2$ and hence [3, Theorem 6.4.16] concludes that $\mathsf{c} \in \mathsf{C}(\omega)$ iff $\mathsf{c}^{-1} \in \ker \mathsf{A}(\omega)$ and the result follows. ∎

We conclude this section describing completely the set $\mathsf{C}(\omega)$ for any $\omega \in \Omega_+(V_n)$.

Proposition 3.4 *Given $\omega \in \Omega_+(V_n)$, we get that*

$$\mathsf{C}(\omega) = \left\{ \left[\Big(\sum_{j=1}^{n-1} x_j^2 q_j(\omega) \Big) \mathsf{c}^{-1}(\omega) + D_{n,n}(\omega) \sum_{j=1}^{n-1} x_j^2 \Big(\mathsf{b}_j(\omega) - \mathsf{b}_{j-1}(\omega) \Big) \right]^{-1} : x_1^2 + \cdots + x_{n-1}^2 > 0 \right\},$$

where $\mathsf{b}_0(\omega) = \mathsf{b}_{n-1}(\omega) = \mathsf{0}$ and for any $j = 1, \ldots, n-2$, the components of $\mathsf{b}_j(\omega)$ are given by

$$b_{jm}(\omega) = \frac{\omega_m \omega_{m+1}}{W_m} \prod_{l=m+1}^{j} \frac{(1 - W_l)}{W_l}, \quad 1 \le m \le j, \qquad b_{jm} = 0, \quad j+1 \le m \le n-1.$$

Proof. From Theorem 3.2 we know that $\mathsf{A}(\omega)$ is a non–singular M–matrix and that $\mathsf{A}^{-1}(\omega) = \mathsf{B}^{-1}(\omega)\mathsf{Q}(\omega)$ where

$$\mathsf{B}^{-1}(\omega) = \begin{bmatrix} 1 & \frac{\omega_1(1-W_2)}{\omega_3 W_1} & \prod\limits_{l=1}^{2} \frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} & \prod\limits_{l=1}^{3} \frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} & \cdots & \frac{\omega_{n-1}}{\omega_n D_{n,n}(\omega)} \prod\limits_{l=1}^{n-2} \frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} \\ 0 & 1 & \frac{\omega_2(1-W_3)}{\omega_4 W_2} & \prod\limits_{l=2}^{3} \frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} & \cdots & \frac{\omega_{n-1}}{\omega_n D_{n,n}(\omega)} \prod\limits_{l=2}^{n-2} \frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} \\ 0 & 0 & 1 & \frac{\omega_3(1-W_4)}{\omega_5 W_3} & \cdots & \frac{\omega_{n-1}}{\omega_n D_{n,n}(\omega)} \prod\limits_{l=3}^{n-2} \frac{\omega_l(1-W_{l+1})}{\omega_{l+2}W_l} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \frac{\omega_{n-1}}{\omega_n D_{n,n}(\omega)} \frac{\omega_{n-2}(1-W_{n-1})}{\omega_n W_{n-2}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\omega_{n-1}}{\omega_n D_{n,n}(\omega)} \end{bmatrix}.$$

Therefore, $\mathsf{b}_j(\omega) = \dfrac{\omega_j \omega_{j+1}}{W_j} \mathsf{B}^{-1}(\omega)\mathsf{e}_j$, $j = 1, \ldots, n-2$, $\mathsf{B}^{-1}(\omega)\mathsf{e}_{n-1} = D_{n,n}^{-1}(\omega)\mathsf{c}^{-1}(\omega)$ which implies that

$$\mathsf{A}^{-1}(\omega)\mathsf{e}_j = q_j(\omega) D_{n,n}^{-1}(\omega)\mathsf{c}^{-1}(\omega) + \mathsf{b}_j(\omega) - \mathsf{b}_{j-1}(\omega), \;\; j = 1, \ldots, n-1,$$

and the result follows. ∎

Acknowledgments. This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under projects MTM2010-19660 and MTM2008-06620-C03-01.

References

[1] R.B. Bapat, S.J. Kirkland, S. Pati, The perturbed Laplacian matrix of a graph, *Linear Multinear Algebra*, **49** (2001) 219–242.

[2] E. Bendito, A. Carmona, A.M. Encinas, M. Mitjana, Generalized inverses of symmetric M-matrices, *Linear Algebra Appl.*, **432** (2010) 2438-2454.

[3] A. Berman, R.J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Classics in Applied Mathematics, vol. 9, SIAM, 1994.

[4] T. Biyikoğlu, J. Leydold , P.F. Stadler, *Laplacian Eigenvectors of Graphs*, LNM 1915, Springer, Berlin, 2007.

[5] Y. Chen, S. J. Kirkland, M. Neumann, Group generalized inverses of M–matrices associated with periodic and nonperiodic Jacobi matrices, *Linear Multilinear Algebra*, **39** (1995) 325–340.

[6] Y. Chen, S. J. Kirkland, M. Neumann, Nonnegative alternating circulants leading to M–matrix group Inverses, *Linear Algebra Appl.*, **233** (1996) 81–97.

[7] Y. Chen, M. Neumann, M–matrix generalized inverses of M–matrices, *Linear Algebra Appl.*, **256** (1997) 263–285.

[8] F. Chung, S.T. Yau, Discrete Green's Functions, *J. Comb. Th. A*, **91** (2000) 191–214.