On the M-matrix inverse problem for singular and symmetric Jacobi matrices

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Statement of the problem

Given $c_1, \ldots, c_{n-1} > 0$, $d_1, \ldots, d_n \geq 0$ and the matrix

$$\mathsf{J} = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-2} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix}$$

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- When $\mathsf{J}^\dagger$ is also an M–matrix?

Schrödinger operators on weighted paths

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- The *combinatorial Laplacian*:

$$\mathcal{L}(u)(x_1) = c_1\big(u(x_1) - u(x_2)\big)$$

$$\mathcal{L}(u)(x_i) = c_{i-1}\big(u(x_i) - u(x_{i-1})\big) + c_i\big(u(x_i) - u(x_{i+1})\big)$$

$$\mathcal{L}(u)(x_n) = c_{n-1}\big(u(x_n) - u(x_{n-1})\big)$$

Schrödinger operators on weighted paths

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The *combinatorial Laplacian*:

- $$\mathsf{L} = \begin{bmatrix} c_1 & -c_1 & & & \\ -c_1 & c_1 + c_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & c_{n-2} + c_{n-1} & -c_{n-1} \\ & & & -c_{n-1} & c_{n-1} \end{bmatrix}$$

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- $$\mathsf{J} = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-1} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix}$$

where $d_i = q_i + c_i + c_{i-1}$

When a Jacobi matrix is an M–matrix?

Given $c_1, \dots, c_n > 0$ and $d_1, \dots, d_n \geq 0$ the matrix

$$J = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-2} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix}$$

is a singular M-matrix iff

$$d_1 = \frac{c_1\omega_2}{\omega_1}, \ d_j = \frac{1}{\omega_j}(c_j\omega_{j+1} + c_{j-1}\omega_{j-1}), \ d_n = \frac{c_{n-1}\omega_{n-1}}{\omega_n}$$

where ω is a weigth: $\omega_i > 0$ and $\omega_1^2 + \dots + \omega_n^2 = 1$

The Moore–Penrose inverse

- $\mathsf{J}\mathsf{J}^\dagger\mathsf{J} = \mathsf{J},\quad \mathsf{J}^\dagger\mathsf{J}\mathsf{J}^\dagger = \mathsf{J}^\dagger$ and $\mathsf{J}\mathsf{J}^\dagger = \mathsf{J}^\dagger\mathsf{J}$

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- $\mathsf{J}^\dagger = \big(g_{ij}\big)$

The Moore–Penrose inverse

- $$g_{ji} = g_{ij} = \omega_i\omega_j \left[\sum_{k=1}^{i-1} \frac{\left(\sum\limits_{l=1}^{k} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} + \sum_{k=j}^{n-1} \frac{\left(\sum\limits_{l=k+1}^{n} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} - \sum_{k=i}^{j-1} \frac{\left(\sum\limits_{l=1}^{k} \omega_l^2\right)\left(\sum\limits_{l=k+1}^{n} \omega_l^2\right)}{c_k\omega_k\omega_{k+1}}\right],$$

  $1 \le i \le j \le n$

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  $1 \le i \le j \le n$
- $g_{ij} < g_{ii+1},\ j = i+2, \dots, n$

When the M–P inverse is an M–matrix?

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- When ω is constant:

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- When ω is constant:

$$\frac{i(n-i)}{c_i} \geq \sum_{k=1}^{i-1} \frac{k^2}{c_k} + \sum_{k=i+1}^{n-1} \frac{(n-k)^2}{c_k}$$

When the M–P inverse is an M–matrix?

- $g_{ii+1} \leq 0$ for any $i = 1, \dots, n-1$

- $$\frac{\Big(\sum\limits_{l=1}^{i} \omega_l^2\Big)\Big(\sum\limits_{l=i+1}^{n} \omega_l^2\Big)}{c_i \omega_i \omega_{i+1}} \geq \sum_{k=1}^{i-1} \frac{\Big(\sum\limits_{l=1}^{k} \omega_l^2\Big)^2}{c_k \omega_k \omega_{k+1}} + \sum_{k=i+1}^{n-1} \frac{\Big(\sum\limits_{l=k+1}^{n} \omega_l^2\Big)^2}{c_k \omega_k \omega_{k+1}}$$

- When ω is constant: **necessarily $n \leq 4$ and**

$$\frac{1}{2} \leq \frac{c_1}{c_2} \leq 2 \text{ when } \underline{n = 3} \text{ or } c_1 = c_3,\ c_2 = 2c_1 \text{ when } \underline{n = 4}$$

[CKN, 95]

Low dimensions: $n = 2$

- $$\frac{\omega_1^2\omega_2^2}{c\,\omega_1\omega_2} \geq 0$$

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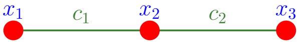
- $$\boxed{\frac{\omega_1^2 \omega_2^2}{c\,\omega_1 \omega_2} \geq 0}$$

- $0 < x < 1$, if $\omega = \left(x, \sqrt{1-x^2}\right)$, we get

$$\mathsf{J} = \begin{bmatrix} \frac{c\sqrt{1-x^2}}{x} & -c \\ -c & \frac{x}{c\sqrt{1-x^2}} \end{bmatrix}$$

$$\mathsf{J}^{\dagger} = \frac{x(1-x^2)}{c} \begin{bmatrix} \sqrt{1-x^2} & -x \\ -x & \frac{x^2}{\sqrt{1-x^2}} \end{bmatrix}$$

Low dimensions: $n = 3$



- $$\frac{\omega_1^2(\omega_2^2+\omega_3^2)}{c_1\omega_1\omega_2} \geq \frac{\omega_3^4}{c_2\omega_2\omega_3}, \quad \frac{(\omega_1^2+\omega_2^2)\omega_3^2}{c_2\omega_2\omega_3} \geq \frac{\omega_1^4}{c_1\omega_1\omega_2}$$

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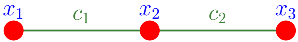

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- J† is an M–matrix iff $\frac{\omega_1^3}{\omega_3(1-\omega_3^2)} \leq \frac{c_1}{c_2} \leq \frac{\omega_1(1-\omega_1^2)}{\omega_3^3}$

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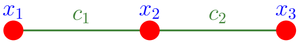


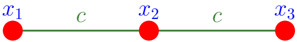
- $\dfrac{\omega_1^2(\omega_2^2+\omega_3^2)}{c_1\omega_1\omega_2} \geq \dfrac{\omega_3^4}{c_2\omega_2\omega_3}, \quad \dfrac{(\omega_1^2+\omega_2^2)\omega_3^2}{c_2\omega_2\omega_3} \geq \dfrac{\omega_1^4}{c_1\omega_1\omega_2}$

- $\mathsf{J}^\dagger$ is an M–matrix iff $\dfrac{\omega_1^3}{\omega_3(1-\omega_3^2)} \leq \dfrac{c_1}{c_2} \leq \dfrac{\omega_1(1-\omega_1^2)}{\omega_3^3}$

- Given $c_1, c_2 > 0$, $\mathsf{J}^\dagger$ is an M–matrix for

$$\left\{\left(\omega_1, \sqrt{1-(1+t^2)\omega_1^2}, t\omega_1\right) : 0 < t < \frac{c_2}{c_1}, \quad 0 < \omega_1 \leq \sqrt{\frac{tc_1}{c_2+t^3c_1}}\right\}$$
$$\cup \left\{\left(\omega_1, \sqrt{1-(1+t^2)\omega_1^2}, t\omega_1\right) : \frac{c_2}{c_1} \leq t, \quad 0 < \omega_1 < \sqrt{\frac{1}{1+t^2}}\right\}$$

An example:



- Taking the values $t = 1$, $0 < x < \dfrac{1}{\sqrt{2}}$

$$\mathsf{J} = \begin{bmatrix} \frac{c\sqrt{1-2x^2}}{x} & -c & 0 \\ -c & \frac{2xc}{\sqrt{1-2x^2}} & -c \\ 0 & -c & \frac{c\sqrt{1-2x^2}}{x} \end{bmatrix}$$

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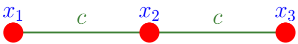

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$$\mathsf{J}^\dagger = \begin{bmatrix} \frac{x(1-2x^2+2x^4)}{c\sqrt{1-2x^2}} & -\frac{x^2(1-2x^2)}{c} & -\frac{2x^3(1-x^2)}{c\sqrt{1-2x^2}} \\ -\frac{x^2(1-2x^2)}{c} & \frac{2x^3\sqrt{1-2x^2}}{c} & -\frac{x^2(1-2x^2)}{c} \\ -\frac{2x^3(1-x^2)}{c\sqrt{1-2x^2}} & -\frac{x^2(1-2x^2)}{c} & \frac{x(1-2x^2+2x^4)}{c\sqrt{1-2x^2}} \end{bmatrix}$$

$n = 4$:

$$\frac{\omega_1^2(\omega_2^2+\omega_3^2+\omega_4^2)}{c_1\omega_1\omega_2} \geq \frac{(\omega_3^2+\omega_4^2)^2}{c_2\omega_2\omega_3} + \frac{\omega_4^4}{c_3\omega_3\omega_4}$$

$$\frac{(\omega_1^2+\omega_2^2)(\omega_3^2+\omega_4^2)}{c_2\omega_2\omega_2} \geq \frac{\omega_1^4}{c_1\omega_1\omega_2} + \frac{\omega_4^4}{c_3\omega_3\omega_4}$$

$$\frac{(\omega_1^2+\omega_2^2+\omega_3^2)\omega_4^2}{c_3\omega_3\omega_4} \geq \frac{\omega_1^4}{c_1\omega_1\omega_2} + \frac{(\omega_1^2+\omega_2^2)^2}{c_2\omega_2\omega_3}$$

Positive

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- $$\frac{\omega_1^2(\omega_2^2+\omega_3^2+\omega_4^2)}{c_1\omega_1\omega_2} \geq \frac{(\omega_3^2+\omega_4^2)^2}{c_2\omega_2\omega_3} + \frac{\omega_4^4}{c_3\omega_3\omega_4}$$
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Positive

- $$\begin{bmatrix} \frac{\omega_1^2(\omega_2^2+\omega_3^2+\omega_4^2)}{\omega_1\omega_2} & -\frac{(\omega_3^2+\omega_4^2)^2}{\omega_2\omega_3} & -\frac{\omega_4^4}{\omega_3\omega_4} \\ -\frac{\omega_1^4}{\omega_1\omega_2} & \frac{(\omega_1^2+\omega_2^2)(\omega_3^2+\omega_4^2)}{\omega_2\omega_2} & -\frac{\omega_4^4}{\omega_3\omega_4} \\ -\frac{\omega_1^4}{\omega_1\omega_2} & -\frac{(\omega_1^2+\omega_2^2)^2}{\omega_2\omega_3} & \frac{(\omega_1^2+\omega_2^2+\omega_3^2)\omega_4^2}{\omega_3\omega_4} \end{bmatrix} \begin{bmatrix} \frac{1}{c_1} \\ \frac{1}{c_2} \\ \frac{1}{c_3} \end{bmatrix} \geq 0$$

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- $$\frac{\omega_1^2(\omega_2^2+\omega_3^2+\omega_4^2)}{c_1\omega_1\omega_2} \geq \frac{(\omega_3^2+\omega_4^2)^2}{c_2\omega_2\omega_3} + \frac{\omega_4^4}{c_3\omega_3\omega_4}$$
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Positive

- $$\underbrace{\begin{bmatrix} \frac{\omega_1^2(\omega_2^2+\omega_3^2+\omega_4^2)}{\omega_1\omega_2} & -\frac{(\omega_3^2+\omega_4^2)^2}{\omega_2\omega_3} & -\frac{\omega_4^4}{\omega_3\omega_4} \\ -\frac{\omega_1^4}{\omega_1\omega_2} & \frac{(\omega_1^2+\omega_2^2)(\omega_3^2+\omega_4^2)}{\omega_2\omega_2} & -\frac{\omega_4^4}{\omega_3\omega_4} \\ -\frac{\omega_1^4}{\omega_1\omega_2} & -\frac{(\omega_1^2+\omega_2^2)^2}{\omega_2\omega_3} & \frac{(\omega_1^2+\omega_2^2+\omega_3^2)\omega_4^2}{\omega_3\omega_4} \end{bmatrix}}_{\mathsf{A}(\omega)} \underbrace{\begin{bmatrix} \frac{1}{c_1} \\ \frac{1}{c_2} \\ \frac{1}{c_3} \end{bmatrix}}_{\mathsf{c}^{-1}} \geq 0$$

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▶ $$\mathsf{A}^{-1}(\omega) = \frac{\omega_2\omega_3}{\omega_2^2\omega_3^3 - \omega_1^2\omega_4^2}\begin{bmatrix} \frac{\omega_3(\omega_1^2+\omega_2^2)}{\omega_1} & \frac{\omega_4^2(1-\omega_4^2)+\omega_3^4}{\omega_1\omega_3} & \frac{\omega_4^2(\omega_3^2+\omega_4^2)}{\omega_1\omega_3} \\ \omega_1^2 & \omega_2^2+\omega_3^2 & \omega_4^2 \\ \frac{\omega_1(\omega_1^2+\omega_2^2)}{\omega_2\omega_4} & \frac{\omega_1^2(1-\omega_1^2)+\omega_2^4}{\omega_2\omega_4} & \frac{\omega_2(\omega_3^2+\omega_4^2)}{\omega_4} \end{bmatrix}$$

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✓ If $\omega_2\omega_3 = \omega_1\omega_4$, $\ker \mathsf{A}(\omega) = \operatorname{span}\{\mathsf{c}^{-1}(\omega)\}$, where

$$\mathsf{c}(\omega) = \left(\frac{\omega_1(\omega_1^2+\omega_2^2)}{\omega_2(\omega_3^2+\omega_4^2)}, \frac{(\omega_1^2+\omega_2^2)}{\omega_2\omega_3}, \frac{\omega_4}{\omega_3} \right)$$

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Example:

- Then, ω is constant

$$\mathsf{J} = \begin{bmatrix} c & -c & 0 & 0 \\ -c & 3c & -2c & 0 \\ 0 & -2c & 3c & -c \\ 0 & 0 & -c & c \end{bmatrix}$$

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Example:

- Then, $\omega = \frac{1}{6}\left(2, 3 \mp \sqrt{5}, 3 \pm \sqrt{5}, 2\right)$

$$\mathsf{J} = \begin{bmatrix} \left(\frac{3\mp\sqrt{5}}{2}\right)c & -c & 0 & 0 \\ -c & (12 \pm 5\sqrt{5})c & -3c & 0 \\ 0 & -3c & (12 \mp 5\sqrt{5})c & -c \\ 0 & 0 & -c & \left(\frac{3\pm\sqrt{5}}{2}\right)c \end{bmatrix}$$

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- Then, $\omega = \frac{1}{\sqrt{3(3+\sqrt{5})}}\left(1, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, 1\right)$

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$$\mathsf{J}^\dagger = \frac{1}{36(47+21\sqrt{5})c}\begin{bmatrix} 591+263\sqrt{5} & -2(20+9\sqrt{5}) & -2(74+33\sqrt{5}) & -(99+43\sqrt{5}) \\ -2(20+9\sqrt{5}) & 177+79\sqrt{5} & -(105+47\sqrt{5}) & -2(74+33\sqrt{5}) \\ -2(74+33\sqrt{5}) & -(105+47\sqrt{5}) & 177+79\sqrt{5} & -2(20+9\sqrt{5}) \\ -(99+43\sqrt{5}) & -2(74+33\sqrt{5}) & -2(20+9\sqrt{5}) & 591+263\sqrt{5} \end{bmatrix}$$

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- For any $n \geq 2$ there exist two families of weights ω and $\widehat{\omega}$ such that $\det \mathsf{A}(\omega) = 0$ and $\det \mathsf{A}(\widehat{\omega}) > 0$