The Green function of a perturbed network

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Abstract

The discrete Green functions and their relationship with discrete Laplace equations have deserved the interest of many researchers using different approaches. In this work we derive the Green function of a perturbed network in terms of the Green function of its base network.

Introduction. Notation

$\Gamma = (V, E, c)$

- $V = \{x_1, \dots, x_n\}$
- $c\colon V \times V \longrightarrow \mathbb{R}$, $c(x_i, x_j) = c_{ij}$
- $\{x_i, x_j\} \in E \Leftrightarrow c_{ij} > 0$

c is the *conductance* of Γ

$\mathcal{C}(V) = \{f\colon V \longrightarrow \mathbb{R}\}$

Projection of u on σ along τ, $\mathcal{P}_{\sigma,\tau}$ is the endomorphism of $\mathcal{C}(V)$

$$\mathcal{P}_{\sigma,\tau}(u) = \sigma\langle\tau, u\rangle$$

The *Laplacian* of Γ is the endomorphism of $\mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x,y)\big(u(x) - u(y)\big),\ x \in V.$$

- $\mathcal{L}$ is singular, self–adjoint and positive semi–definite on $\mathcal{C}(V)$
- $\mathcal{L}(u) = 0$ iff u is a constant function
- $\mathcal{G}$ is the *orthogonal Green operator*

Laplacian operators associated with a perturbation of the conductance

- $\epsilon\colon V \times V \longrightarrow [0, +\infty)$ a symmetric function
- the perturbed conductance $c + \epsilon$
- $\mathcal{L}^\epsilon$ the perturbed combinatorial Laplacian
- $\mathcal{G}^\epsilon$ its orthogonal Green operator

For any $x, y \in V$, we consider

- The *dipole* $\sigma_{xy} = \sqrt{\dfrac{\epsilon(x,y)}{2}}(\varepsilon_x - \varepsilon_y)$
- The *effective resistance* $R(x,y)$ is the only solution of $\mathcal{L}(u) = \sigma_{xy}$
- $R(x,y,z,t) = \frac{1}{2}\Big(R(x,t) - R(t,y) + R(y,z) - R(z,x)\Big)$

Relation between the effective resistance and the Green

$$R(x,y) = G(x,x) + G(y,y) - 2G(x,y)$$

which implies that $R\colon V \times V \longrightarrow [0, +\infty)$ is symmetric and vanishes on the diagonal

Main result

It is satisfied that $\mathcal{L}^\epsilon = \mathcal{L} + \sum_{x,y\in V} \mathcal{P}_{\sigma_{xy}}$ and $\mathcal{G}^\epsilon = \mathcal{G} - \sum_{x,y,z,t\in V} b(xy,zt)\mathcal{P}_{\mathcal{G}(\sigma_{xy}),\mathcal{G}(\sigma_{zt})}$, where $\big(b(xy,zt)\big) = \Big[\mathrm{I} + \big(\langle\mathcal{G}(\sigma_{xy}), \sigma_{zt}\rangle\big)\Big]^{-1}$

In particular, if we add only one edge of conductance $\epsilon > 0$ between vertices x and y and consider the dipole $\sigma = \sqrt{\epsilon}(\varepsilon_x - \varepsilon_y)$. Then,

$$\mathcal{G}^\epsilon = \mathcal{G} - \frac{1}{1 + \epsilon R(x,y)}\mathcal{P}_{\mathcal{G}(\sigma)} \qquad R^\epsilon(v,w) = R(v,w) - \frac{\epsilon R(x,y,v,w)^2}{1 + \epsilon R(x,y)} \qquad \text{and} \qquad R^\epsilon(x,y) = \frac{1}{\epsilon + \frac{1}{R(x,y)}}$$

The Green function of a weighted 9–path

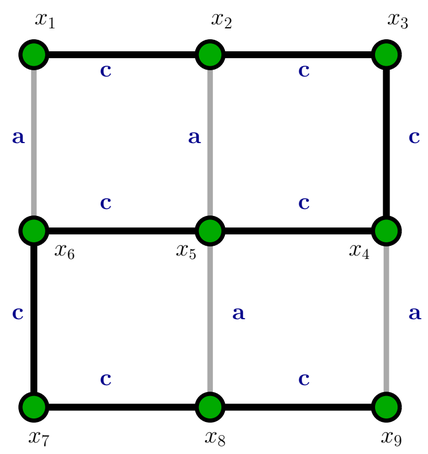
Consider the 3×3–lattice. The conductance function is:

- $c(x_i, x_{i+1}) = c > 0$, for $i = 1, \dots, 8$.
- $c(x_1, x_6) = c(x_2, x_5) = c(x_5, x_8) = c(x_4, x_9) = a > 0$.

The Hamiltonian path (thick line) has constant conductance c and its Green function is, see [1]

$$G(x_i, x_j) = \frac{1}{54c}\Big[190 + 3\big[i(i-10) + j(j-10) - 9|i-j|\big]\Big]$$

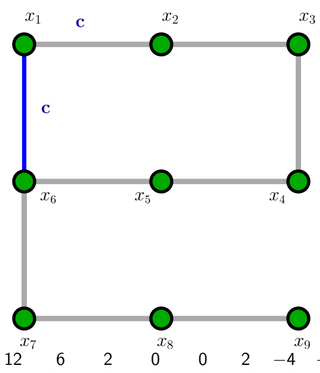
$i, j = 1, \dots, 9$



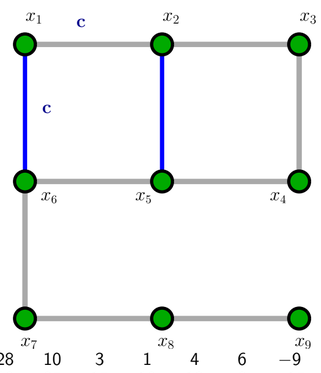
The matrix associated to $\mathcal{G}$ is,

$$\mathrm{G} = \frac{1}{27c}\begin{bmatrix} 68 & 44 & 23 & 5 & -10 & -22 & -31 & -37 & -40 \\ 44 & 47 & 26 & 8 & -7 & -19 & -28 & -34 & -37 \\ 23 & 26 & 32 & 14 & -1 & -13 & -22 & -28 & -31 \\ 5 & 8 & 14 & 23 & 8 & -4 & -13 & -19 & -22 \\ -10 & -7 & -1 & 8 & 20 & 8 & -1 & -7 & -10 \\ -22 & -19 & -13 & -4 & 8 & 23 & 14 & 8 & 5 \\ -31 & -28 & -22 & -13 & -1 & 14 & 32 & 26 & 23 \\ -37 & -34 & -28 & -19 & -7 & 8 & 26 & 47 & 44 \\ -40 & -37 & -31 & -22 & -10 & 5 & 23 & 44 & 68 \end{bmatrix}$$

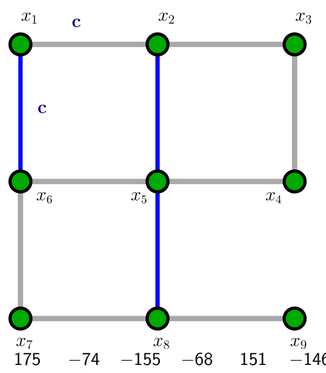
From a 9–path to a 3×3–lattice



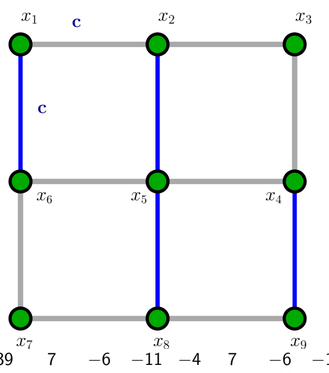
$$\mathrm{G}_1 = \frac{1}{18c}\begin{bmatrix} 12 & 6 & 2 & 0 & 0 & 2 & -4 & -8 & -10 \\ 6 & 15 & 8 & 3 & 0 & -1 & -7 & -11 & -13 \\ 2 & 8 & 16 & 8 & 2 & -2 & -8 & -12 & -14 \\ 0 & 3 & 8 & 15 & 6 & -1 & -7 & -11 & -13 \\ 0 & 0 & 2 & 6 & 12 & 2 & -4 & -8 & -10 \\ 2 & -1 & -2 & -1 & 2 & 7 & 1 & -3 & -5 \\ -4 & -7 & -8 & -7 & -4 & 1 & 13 & 9 & 7 \\ -8 & -11 & -12 & -11 & -8 & -3 & 9 & 23 & 21 \\ -10 & -13 & -14 & -13 & -10 & -5 & 7 & 21 & 37 \end{bmatrix}$$



$$\mathrm{G}_2 = \frac{1}{45c}\begin{bmatrix} 28 & 10 & 3 & 1 & 4 & 6 & -9 & -19 & -24 \\ 10 & 25 & 15 & 10 & 10 & 0 & -15 & -25 & -30 \\ 3 & 15 & 38 & 21 & 9 & -4 & -19 & -29 & -34 \\ 1 & 10 & 21 & 37 & 13 & -3 & -18 & -28 & -33 \\ 4 & 10 & 9 & 13 & 22 & 3 & -12 & -22 & -27 \\ 6 & 0 & -4 & -3 & 3 & 17 & 2 & -8 & -13 \\ -9 & -15 & -19 & -18 & -12 & 2 & 32 & 22 & 17 \\ -19 & -25 & -29 & -28 & -22 & -8 & 22 & 57 & 52 \\ -24 & -30 & -34 & -33 & -27 & -13 & 17 & 52 & 92 \end{bmatrix}$$



$$\mathrm{G}_3 = \frac{1}{1512c}\begin{bmatrix} 835 & 175 & -74 & -155 & -68 & 151 & -146 & -275 & -443 \\ 175 & 595 & 238 & 49 & 28 & -77 & -266 & -287 & -455 \\ -74 & 238 & 988 & 394 & -32 & -218 & -380 & -374 & -542 \\ -155 & 49 & 394 & 907 & 76 & -191 & -326 & -293 & -461 \\ -68 & 28 & -32 & 76 & 352 & 4 & -104 & -44 & -212 \\ 151 & -77 & -218 & -191 & 4 & 547 & 142 & -95 & -263 \\ -146 & -266 & -380 & -326 & -104 & 142 & 844 & 202 & 34 \\ -275 & -287 & -374 & -293 & -44 & -95 & 202 & 667 & 499 \\ -443 & -455 & -542 & -461 & -212 & -263 & 34 & 499 & 1843 \end{bmatrix}$$



$$\mathrm{G}^\epsilon = \frac{1}{72c}\begin{bmatrix} 39 & 7 & -6 & -11 & -4 & 7 & -6 & -11 & -15 \\ 7 & 26 & 7 & -4 & 0 & -4 & -11 & -10 & -11 \\ -6 & 7 & 39 & 7 & -4 & -11 & -15 & -11 & -6 \\ -11 & -4 & 7 & 26 & 0 & -10 & -11 & -4 & 7 \\ -4 & 0 & -4 & 0 & 16 & 0 & -4 & 0 & -4 \\ 7 & -4 & -11 & -10 & 0 & 26 & 7 & -4 & -11 \\ -6 & -11 & -15 & -11 & -4 & 7 & 39 & 7 & -6 \\ -11 & -10 & -11 & -4 & 0 & -4 & 7 & 26 & 7 \\ -15 & -11 & -6 & 7 & -4 & -11 & -6 & 7 & 39 \end{bmatrix}$$

References

E. Bendito, A.Carmona, A.M. Encinas, M. Mitjana,
The M-matrix inverse problem for singular and symmetric Jacobi matrices,
LAA, **436** (2012), 1090–1098.

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