

Discrete Serrin's Problem

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Real Sociedad Matemática Española

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The original Serrin's Problem

Given $\Omega \subset \mathbb{R}^n$, with smooth boundary $\delta(\Omega)$, if u is the unique solution of

$$\begin{aligned} -\Delta(u) &= 1 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \delta(\Omega) \end{aligned}$$

then $\frac{\partial u}{\partial n}$ is constant iff Ω is a ball and $u(x) = \frac{1}{2n}(R^2 - |x|^2)$;
that is, u is radial.

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- ▶ Moving planes
- ▶ Minimum principle and Green Identities

Discrete Serrin's Problem

Given $\Gamma = (F \cup \delta(F), c)$ a network with boundary if u is the unique solution of

$$\mathcal{L}(u) = 1 \quad \text{on } F$$

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then if $\frac{\partial u}{\partial n}$ is constant, what can we say about Γ and u ?

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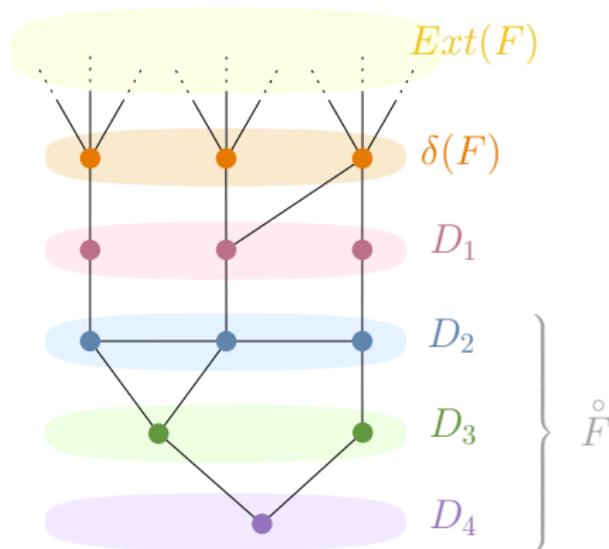
- ▶ Minimum principle and Green Identities
- ▶ Existence of equilibrium measure

Network Topology

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- ▶ Given $F \subset V$ consider the sets



$$r(F) = \max_{x \in F} \{d(x, \delta(F))\}$$

Operators

► **Combinatorial Laplacian** $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y)) = k(x)u(x) - \sum_{y \in V} c(x, y)u(y)$$

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► **Matrix version**

$$\begin{pmatrix} k(x_1) & -c(x_1, x_2) & \cdots & -c(x_1, x_n) \\ -c(x_1, x_2) & k(x_2) & \cdots & -c(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -c(x_1, x_n) & -c(x_2, x_n) & \cdots & k(x_n) \end{pmatrix} = D - A$$

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► **Normal derivative:** $u \in \mathcal{C}(V)$ and F connected proper set

$$\frac{\partial u}{\partial \mathbf{n}}(x) = \sum_{y \in F} c(x, y)(u(x) - u(y)), \quad \text{for any } x \in \delta(F)$$

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Gauss Theorem:
$$\sum_{x \in F} \mathcal{L}(u)(x) = - \sum_{x \in \delta(F)} \frac{\partial u}{\partial \mathbf{n}}(x)$$

Minimum principle

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If $u \in \mathcal{C}(V)$ is superharmonic on F , then

▶
$$\min_{x \in \delta(F)} \{u(x)\} \leq \min_{x \in F} \{u(x)\}$$

The equality holds iff $u = a\chi_{\bar{F}}$

Generalized minimum principles

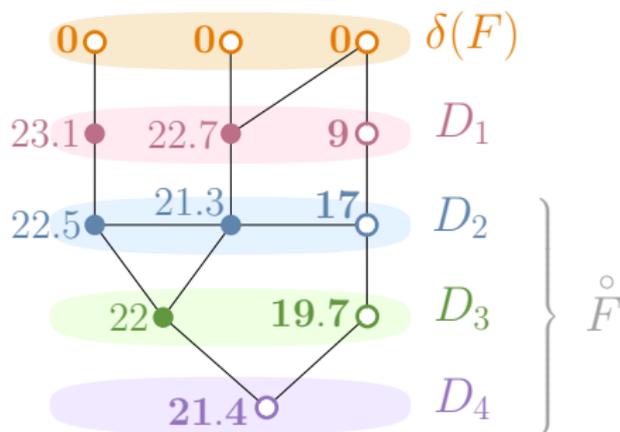
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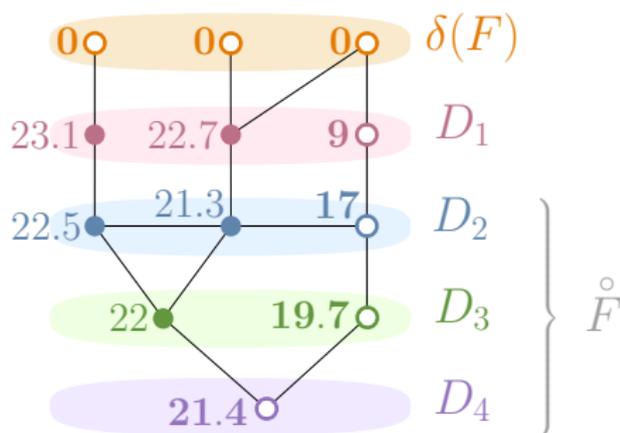
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▶ If $u \in \mathcal{C}^+(F)$ is a strictly superharmonic function on F , then for any $x \in F$ **there exists** $y \in \bar{F}$ such that $c(x, y) > 0$ and $u(y) < u(x)$

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Level sets

- ▶ Given $u \in \mathcal{C}^+(F)$ we denote $0 = u_0 < u_1 < \dots < u_s$
- ▶ **Level set** $U_i = \{x \in F : u(x) = u_i\}$

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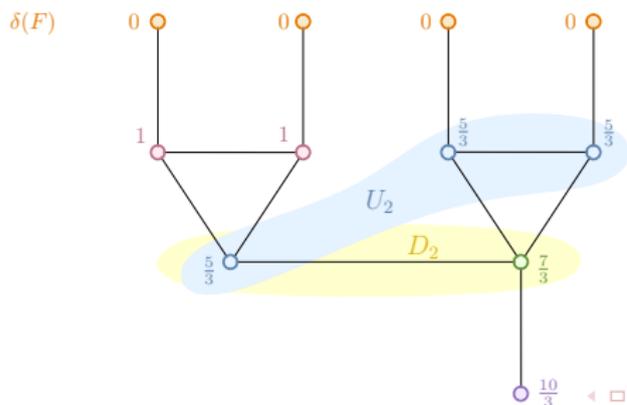
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▶ If $u \in \mathcal{C}^+(F)$ is a strictly superharmonic function on F satisfying $U_j = D_j$ for all $j = 0, \dots, i$, then $U_{i+1} \subset D_{i+1}$

Radial Functions

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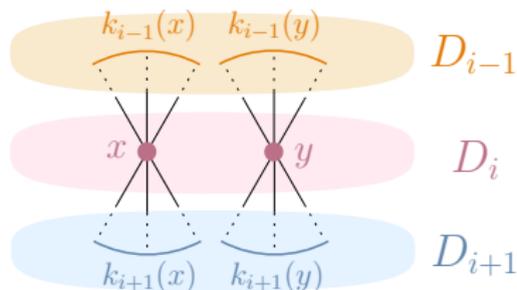
- ▶
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Moreover, for any $x \in D_0$,

- ▶
$$\frac{\partial u}{\partial n}(x) = -k_1(x)u_1 < 0$$

Serrin's problem

Given $\Gamma = (F \cup \delta(F), c)$ a network with boundary if ν is the equilibrium measure of F

$$\mathcal{L}(\nu) = 1 \text{ on } F, \quad \nu = 0 \text{ on } \delta(F)$$

then if $\frac{\partial \nu}{\partial n} = C$, what can we say about Γ and ν ?

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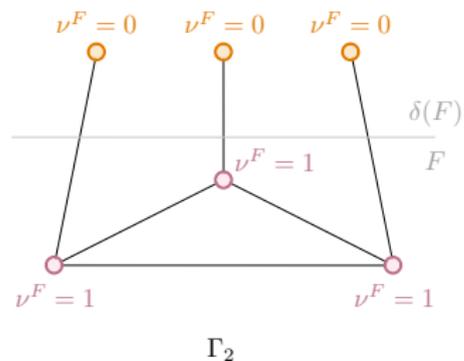
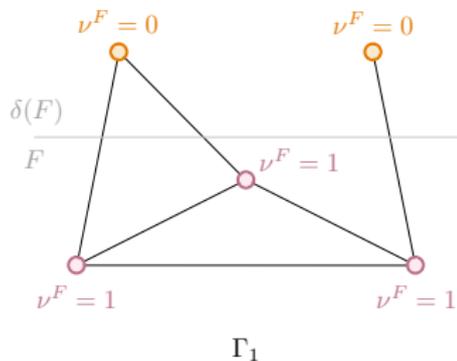
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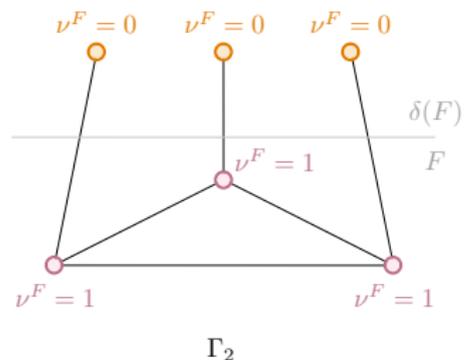
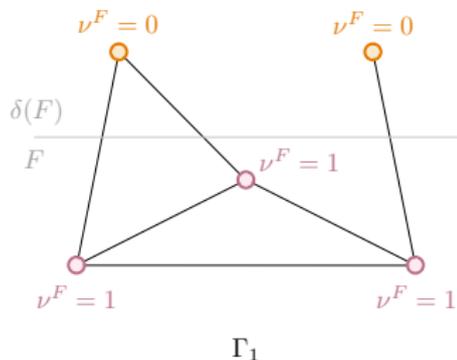
- ▶ Has Γ **ball-like structure**?
- ▶ Is ν **radial**?

- ▶ If ν satisfies Serrin's condition, then $C = -\frac{|F|}{|\delta(F)|}$

Separated boundary

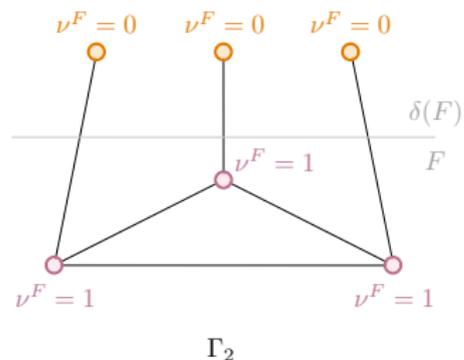
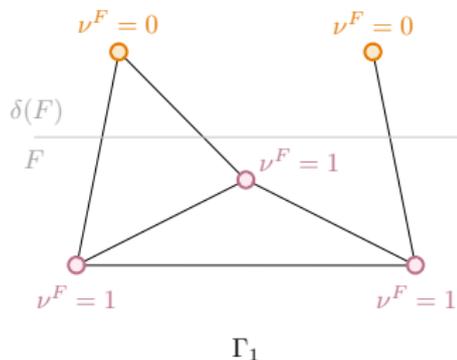


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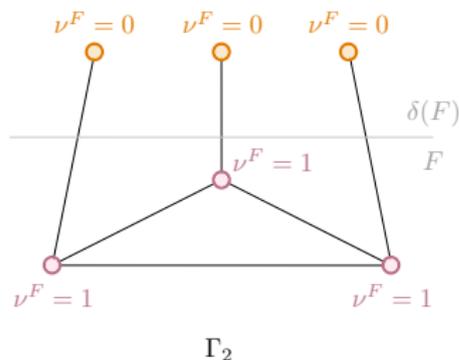
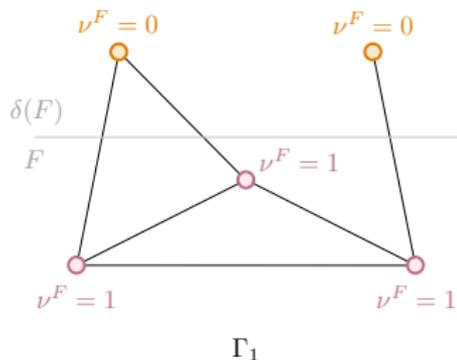
► For any $x \in \delta(F)$ there $\exists!$ $\hat{x} \in D_1$ such that $c(x, \hat{x}) > 0$

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- ▶ We suppose that $|F| \geq 2$

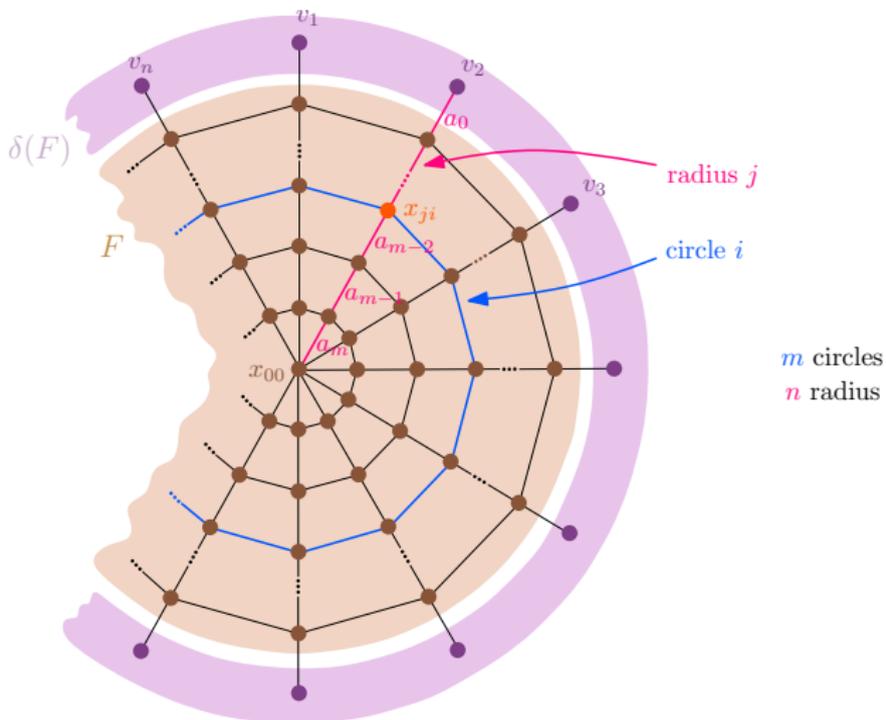
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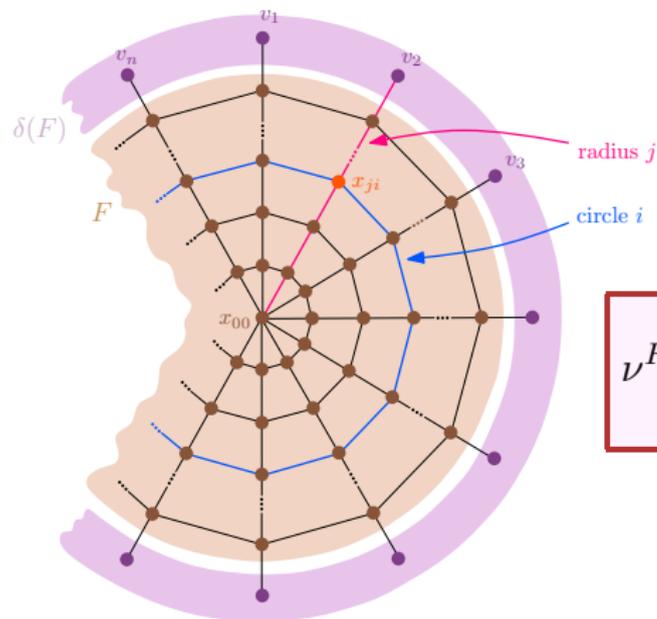
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- ▶ We suppose that $|F| \geq 2$

- ▶ If ν satisfies Serrin's condition, then $U_1 = D_1$ iff $c(x, \hat{x})$ is constant. Therefore, $U_2 \subset D_2$

Spider network with radial conductances

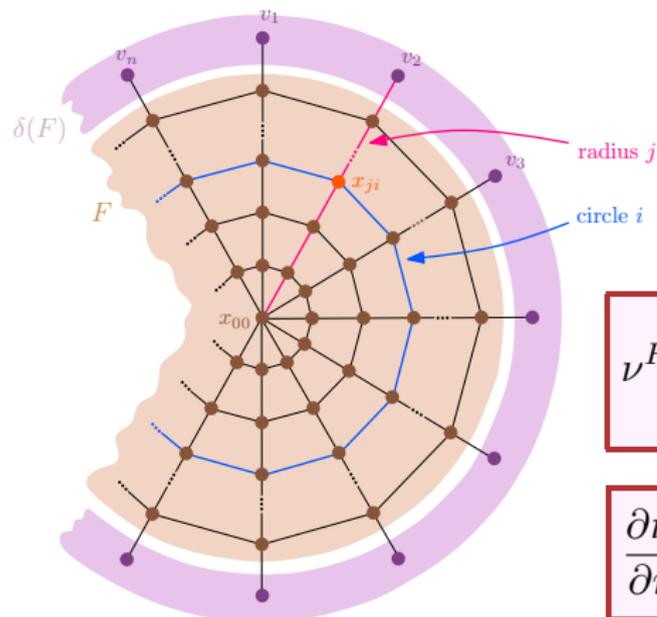


Spider network with radial conductances



$$\nu^F(x_{js}) = \frac{1}{n} \sum_{i=0}^{m-s} \frac{n(m-i) + 1}{a_i}$$

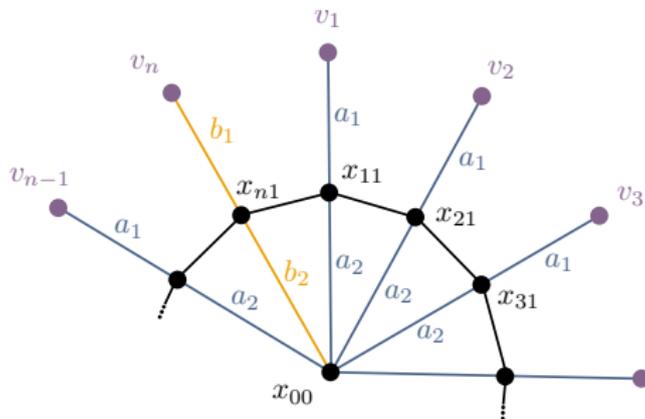
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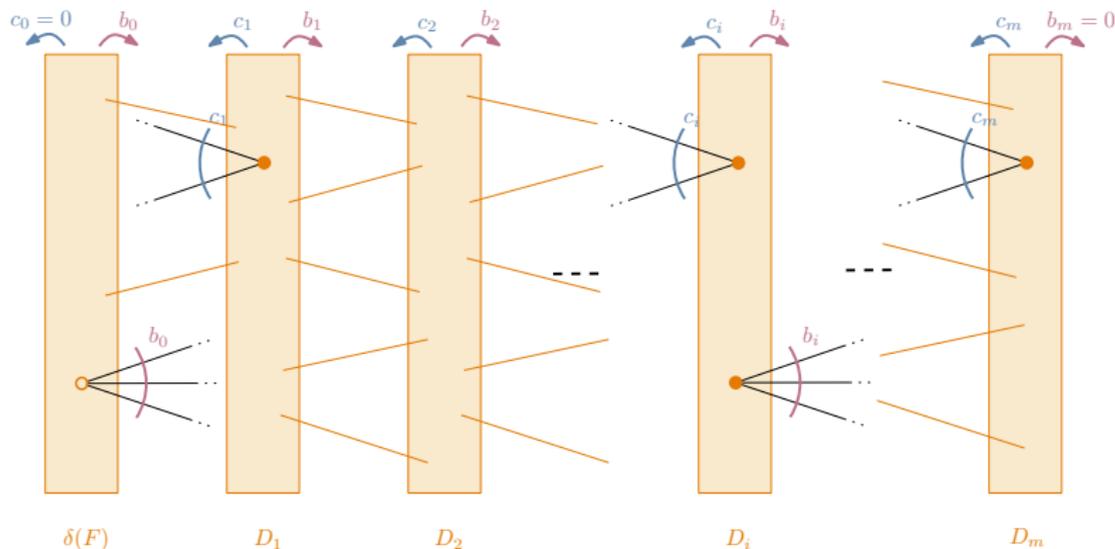
$$\frac{\partial \nu^F}{\partial \eta_F} = -\frac{(nm+1)}{n}$$

Spider network



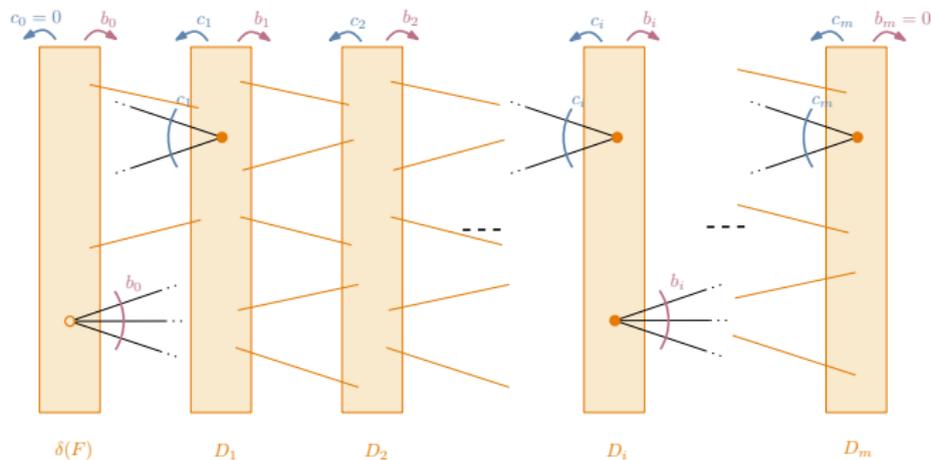
► Serrin's condition holds iff $b_1 = a_1$ and $b_2 = a_2$

Regular Layered Networks



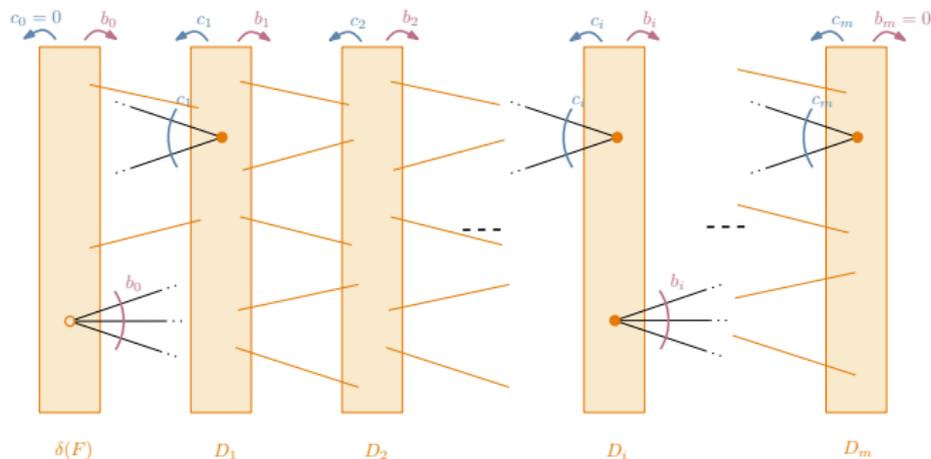
- For any $x \in D_i$, $k_{i-1}(x) = c_i$ and $k_{i+1}(x) = b_i$, $i = 1, \dots, m$

Regular Layered Networks



$$\blacktriangleright \nu^F(x) = \sum_{j=1}^s \frac{1}{b_{j-1}} \sum_{k=j}^m \left(\prod_{\ell=j-1}^{k-1} \frac{b_\ell}{c_{\ell+1}} \right) \quad \text{for all } x \in D_s$$

Regular Layered Networks



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Characterization

Let Γ be a network such that for all $i = 1, \dots, m - 1$,

$$k_{i+1}(x) + k_{i-1}(x) = d_i \quad \text{for all } x \in D_i$$

$$U_i = D_i$$

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▶ Then, ν satisfies Serrin's condition iff Γ is a layered regular graph