Perturbations of Discrete Elliptic Operators

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The setting

- Finite connected network: $\Gamma = (V, c)$, $c\colon V \times V \longrightarrow [0, +\infty)$
- Space of real functions: $\mathcal{C}(V) = \{u\colon V \longrightarrow \mathbb{R}\}$
- Inner product on $\mathcal{C}(V)$: $\langle u, v\rangle = \sum_{x\in V} u(x)v(x)$
- Weights: $\Omega(V) = \{\omega \in \mathcal{C}(V) : \omega(x) \neq 0,\ x \in V,\ \langle \omega, \omega\rangle = 1\}$
- Positive weights: $\Omega_+(V) = \{\omega \in \Omega(V) : \omega(x) > 0,\ x \in V\}$
- Dirac functions: ε_x
- Linear operators: $\mathcal{K}\colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$
- Kernel: $K \in \mathcal{C}(V \times V) \Longrightarrow \mathcal{K}(u)(x) = \sum_{y\in V} K(x, y)u(y)$

Schrödinger–like Operators

Consider $\mathcal{F}\colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ given by

$$\mathcal{F}(u)(x) = \sum_{y\in V} c(x,y)\big(u(x) \pm u(y)\big)$$

and $\mathcal{F}_q(u) = \mathcal{F}(u) + qu$, where $q \in \mathcal{C}(V)$ is called potential

- **Matrix version:** If $k(x) = \sum_{y\in V} c(x,y)$ (generalized degree)

$$\begin{pmatrix} k(x_1)+q(x_1) & \pm c(x_1,x_2) & \cdots & \pm c(x_1,x_n) \\ \pm c(x_1,x_2) & k(x_2)+q(x_2) & \cdots & \pm c(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \pm c(x_1,x_n) & \pm c(x_2,x_n) & \cdots & k(x_n)+q(x_n) \end{pmatrix}$$

Schrödinger–like Operators

Consider $\mathcal{L}\colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x,y)\big(u(x)-u(y)\big)$$

Combinatorial Laplacian

- **Matrix version:**

$$\begin{pmatrix} k(x_1) & -c(x_1,x_2) & \cdots & -c(x_1,x_n) \\ -c(x_1,x_2) & k(x_2) & \cdots & -c(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -c(x_1,x_n) & -c(x_2,x_n) & \cdots & k(x_n) \end{pmatrix} = \mathsf{D}-\mathsf{A}$$

Schrödinger–like Operators

Consider $\mathcal{Q}\colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{Q}(u)(x) = \sum_{y\in V} c(x,y)\big(u(x)+u(y)\big)$$

Signless Laplacian

- **Matrix version:**

$$\begin{pmatrix} k(x_1) & +c(x_1,x_2) & \cdots & +c(x_1,x_n) \\ +c(x_1,x_2) & k(x_2) & \cdots & +c(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ +c(x_1,x_n) & +c(x_2,x_n) & \cdots & k(x_n) \end{pmatrix} = \mathsf{D}+\mathsf{A}$$

Singular Schrödinger–like Operators

Combinatorial Laplacian	Signless Laplacian
$\mathcal{L}(u)(x) = \sum_{y\in V} c(x,y)\big(u(x)-u(y)\big)$ $\Downarrow$ $\mathcal{L}(u)=0 \Leftrightarrow u = a\cdot 1$	$\mathcal{Q}(u)(x) = \sum_{y\in V} c(x,y)\big(u(x)+u(y)\big)$ $\Downarrow$ singular $\Leftrightarrow \Gamma$ is (V_0,V_1)–bipartite

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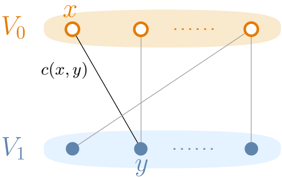

$\Longrightarrow$

$$\mathcal{Q}(u)=0 \Leftrightarrow u = a\cdot\big(\chi_{V_0}-\chi_{V_1}\big)$$

Elliptic Schrödinger–like Operators

Consider $\mathcal{F}\colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ and $q \in \mathcal{C}(V)$, where

$$\mathcal{F}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) \pm u(y)\big)$$

and the Schrödinger operator $\mathcal{F}_q(u) = \mathcal{F}(u) + qu$

- $\mathcal{F}_q$ is elliptic iff it is positive semidefinite and its lowest eigenvalue, λ, is simple
- If $\omega \in \Omega(V)$ is s.t. $\mathcal{F}_q(\omega) = \lambda\omega$, then $\mathcal{F}_q$ is (λ,ω)–elliptic

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- $\mathcal{L}$ is $(0,\omega)$–elliptic, where $\omega = \sqrt{\frac{1}{n}}\,1$

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- If $\omega \in \Omega(V)$ is s.t. $\mathcal{F}_q(\omega) = \lambda\omega$, then $\mathcal{F}_q$ is (λ,ω)–elliptic
- If Γ is (V_0, V_1)–bipartite, then $\mathcal{Q}$ is $(0,\omega)$–elliptic, where $\omega = \sqrt{\dfrac{1}{n}}\,(\chi_{V_0} - \chi_{V_1})$

Doob Transform

Consider the Schrödinger–like operator

$$\mathcal{F}_q(u)(x) = \sum_{y\in V} c(x,y)\big(u(x) \pm u(y)\big) + q(x)u(x)$$

- Potential associated with $\sigma \in \Omega_+(V)$: $q_\sigma = -\sigma^{-1}\mathcal{L}(\sigma)$

- $$\mathcal{F}_q(u)(x) = \frac{1}{\sigma(x)} \sum_{y\in V} c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)} \pm \frac{u(y)}{\sigma(y)}\right) + \big(q(x) - q_\sigma(x)\big)u(x)$$

Doob Transform

Consider the Schrödinger–like operator

$$\mathcal{F}_q(u)(x) = \sum_{y\in V} c(x,y)\big(u(x) \pm u(y)\big) + q(x)u(x)$$

- Potential associated with $\sigma \in \Omega_+(V)$: $q_\sigma = -\sigma^{-1}\mathcal{L}(\sigma)$
- $$\mathcal{F}_q(u)(x) = \frac{1}{\sigma(x)} \sum_{y\in V} c(x,y)\sigma(x)\sigma(y)\left(\frac{u(x)}{\sigma(x)} \pm \frac{u(y)}{\sigma(y)}\right) + \big(q(x) - q_\sigma(x)\big)u(x)$$
- $q - q_\sigma = \lambda \geq 0 \Rightarrow \mathcal{F}_q$ is positive semidefinite

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Elliptic?

Doob Transform

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- Potential associated with $\sigma \in \Omega_+(V)$: $q_\sigma = -\sigma^{-1}\mathcal{L}(\sigma)$
- $$\mathcal{F}_q(u)(x) = \frac{1}{\sigma(x)} \sum_{y \in V} c(x,y)\sigma(x)\sigma(y) \left(\frac{u(x)}{\sigma(x)} \pm \frac{u(y)}{\sigma(y)}\right) + \big(q(x) - q_\sigma(x)\big)u(x)$$
- $\mathcal{L}_q$ is elliptic iff $q = q_\sigma + \lambda$, $\lambda \geq 0$ [BCE 2005] 

Doob Transform and Ellipticity

Let $q = q_\sigma + \lambda$, $\sigma \in \Omega_+(V)$, $\lambda \geq 0$ and $\mathcal{F}, \widehat{\mathcal{F}} \colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$,

$$\mathcal{F}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) \pm u(y)\big)$$

$$\widehat{\mathcal{F}}(u)(x) = \sum_{y \in V} c(x,y)\sigma(x)\sigma(y)\big(u(x) \pm u(y)\big)$$

- Define $\widehat{c}(x,y) = c(x,y)\sigma(x)\sigma(y)$ and consider $\widehat{\Gamma} = (V, \widehat{c})$

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- Define $\widehat{c}(x,y) = c(x,y)\sigma(x)\sigma(y)$ and consider $\widehat{\Gamma} = (V, \widehat{c})$
- $$\mathcal{F}_q(u) = \frac{1}{\sigma}\widehat{\mathcal{F}}\Big(\frac{u}{\sigma}\Big) + \lambda u \Rightarrow \langle \mathcal{F}_q(u), u\rangle = \Big\langle \widehat{\mathcal{F}}\Big(\frac{u}{\sigma}\Big), \frac{u}{\sigma}\Big\rangle + \lambda\langle u, u\rangle$$

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- Define $\widehat{c}(x,y)=c(x,y)\sigma(x)\sigma(y)$ and consider $\widehat{\Gamma}=(V,\widehat{c})$
- $$\mathcal{F}_q(u)=\frac{1}{\sigma}\widehat{\mathcal{F}}\left(\frac{u}{\sigma}\right)+\lambda u\Longrightarrow\langle\mathcal{F}_q(u),u\rangle\geq\lambda\langle u,u\rangle$$

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- $$\mathcal{F}_q(u) = \frac{1}{\sigma}\widehat{\mathcal{F}}\Big(\frac{u}{\sigma}\Big) + \lambda u \Longrightarrow \langle \mathcal{F}_q(u), u\rangle \geq \lambda \langle u, u\rangle$$
- $\mathcal{F}_q$ is (λ, ω)–elliptic iff $\widehat{\mathcal{F}}$ is $(0, \sigma^{-1}\omega)$–elliptic

Doob Transform and Signless Laplacian

Let $q = q_\sigma + \lambda$, $\sigma \in \Omega_+(V)$, $\lambda \geq 0$ and

$$\mathcal{Q}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) + u(y)\big)$$

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- Assume that Γ is (V_0, V_1)–bipartite$\Rightarrow$ $\widehat{\Gamma}$ is (V_0, V_1)–bipartite
- $\mathcal{Q}_q$ is (λ, ω)–elliptic where $\omega = \sigma\big(\chi_{V_0} - \chi_{V_1}\big)$

Orthogonal Green Function

Consider $\mathcal{F}\colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ a (λ,ω)–elliptic operator

- $\mathcal{F}$ determines an automorphism on $\omega^{\perp}$
- Orthogonal Green Operator: $\mathcal{G}(u) = \mathcal{F}^{-1}\big(u - \langle \omega, u\rangle \omega\big)$
- $\mathcal{G}$ is $(0,\omega)$–elliptic
- Orthogonal Green Function: G, the kernel of $\mathcal{G}$
- $G(x,y) = G(y,x)$ and $G(x,x) > 0$

Resistances and Kirchhoff Index

Consider $\mathcal{F}$ a (λ,ω)–elliptic operator, $\mathcal{G}$ its orthogonal Green operator and G the orthogonal Green Function

- ω–Dipole between $x,y\in V$: $\tau_{xy}=\dfrac{\varepsilon_x}{\omega(x)}-\dfrac{\varepsilon_y}{\omega(y)}\in\omega^\perp$
- Effective Resistance between $x,y\in V$: $R(x,y)=\langle\mathcal{G}(\tau_{xy}),\tau_{xy}\rangle$
- $R(x,y)=R(y,x)$, $R(x,y)\ge 0$ and $R(x,y)=0$ iff $x=y$

  $$R(x,y)=\frac{G(x,x)}{\omega(x)^2}+\frac{G(y,y)}{\omega(y)^2}-\frac{2G(x,y)}{\omega(x)\omega(y)}$$
- Kirchhoff Index of $\mathcal{F}$: $\mathsf{K}=\dfrac{1}{2}\displaystyle\sum_{x,y\in V}R(x,y)\omega(x)^2\omega(y)^2$

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- Kirchhoff Index of $\mathcal{F}$: $\mathsf{K}=\sum_{x\in V}G(x,x)$

Perturbations of Elliptic Signless Laplacian

Assume that Γ is (V_0, V_1)–bipartite and consider

$$\mathcal{Q}_q(u)(x) = \sum_{y\in V} c(x,y)\big(u(x)+u(y)\big) + q(x)u(x)$$

where $q = q_\sigma + \lambda \Rightarrow \mathcal{Q}_q$ is (λ,ω)–elliptic, $\omega = \sigma\big(\chi_{V_0} - \chi_{V_1}\big)$

- Consider $\epsilon \colon V\times V \longrightarrow [0,+\infty)$ and $\chi = \chi_{(V_0\times V_1)\cup(V_1\times V_0)}$

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- Consider $\epsilon\colon V\times V \longrightarrow [0,+\infty)$ and $\chi = \chi_{(V_0\times V_1)\cup(V_1\times V_0)}$
- $$\mathcal{Q}^\epsilon(u)(x) = u(x)\sum_{y\in V}(c+\boldsymbol{\epsilon})(x,y) + \sum_{y\in V}(\boldsymbol{\epsilon}\cdot\boldsymbol{\chi})(x,y)\,u(y) - \sum_{y\in V}\big(\boldsymbol{\epsilon}\cdot(\mathbf{1}-\boldsymbol{\chi})\big)(x,y)\,u(y) + \boldsymbol{q}^\epsilon(x)u(x)$$
  $$\boldsymbol{q}^\epsilon(x) = \lambda - \frac{1}{\sigma(x)}\sum_{y\in V}(c+\boldsymbol{\epsilon})(x,y)\big(\sigma(x)-\sigma(y)\big),$$ is (λ,ω)–elliptic

Perturbations of Elliptic Signless Laplacian

- $\rho_{xy} = \sqrt{\epsilon(x,y)\sigma(x)\sigma(y)}$
- $\mathsf{v}_x = \Big(\frac{1}{2}\rho_{z\hat{z}}\big[R(z,x) - R(\hat{z},x)\big]\Big) \in \mathbb{R}^{n^2}, \quad \mathsf{r} = \sum_{t\in V} \mathsf{v}_t\omega(t)^2$
- $\mathsf{M} = \mathsf{I} + \Big(\rho_{xy}\rho_{\hat{x}\hat{y}}\big\langle\mathcal{G}(\tau_{xy}), \tau_{\hat{x}\hat{y}}\big\rangle\Big)$ is positive definite

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- $\mathsf{M} = \mathsf{I} + \Big(\rho_{xy}\rho_{\hat{x}\hat{y}}\langle \mathcal{G}(\tau_{xy}), \tau_{\hat{x}\hat{y}}\rangle\Big)$ is positive definite
- $G^{\epsilon}(x,y) = G(x,y) - \omega(x)\omega(y)\langle \mathsf{M}^{-1}(\mathsf{r}-\mathsf{v}_x), \mathsf{r}-\mathsf{v}_y\rangle$

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- $\mathsf{M} = \mathsf{I} + \left(\rho_{xy}\rho_{\hat{x}\hat{y}}\langle \mathcal{G}(\tau_{xy}), \tau_{\hat{x}\hat{y}}\rangle\right)$ is positive definite
- $G^\epsilon(x,y) = G(x,y) - \omega(x)\omega(y)\langle \mathsf{M}^{-1}(\mathsf{r} - \mathsf{v}_x), \mathsf{r} - \mathsf{v}_y\rangle$
- $R^\epsilon(x,y) = R(x,y) - \langle \mathsf{M}^{-1}(\mathsf{v}_x - \mathsf{v}_y), \mathsf{v}_x - \mathsf{v}_y\rangle$

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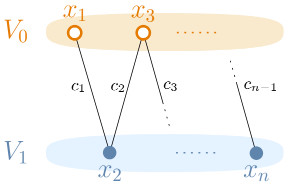
- $R^{\epsilon}(x,y) = R(x,y) - \langle \mathsf{M}^{-1}(\mathsf{v}_x-\mathsf{v}_y), \mathsf{v}_x-\mathsf{v}_y\rangle$
- $\mathsf{K}^{\epsilon} = \mathsf{K} + \langle \mathsf{M}^{-1}\mathsf{r}, \mathsf{r}\rangle - \sum_{z\in V}\langle \mathsf{M}^{-1}\mathsf{v}_z, \mathsf{v}_z\rangle \omega(z)^2$

Example: A path

Consider $\mathcal{Q}$ the signless laplacian of the path on n vertices with conductances $c_1, \dots, c_{n-1} > 0$, $\sigma \in \Omega_+(V)$ and $\mathcal{Q}_{q_\sigma}$



- If $\omega(x_j) = w_j = (-1)^j \sigma(x_j) = (-1)^j \sigma_j$, $\mathcal{Q}_{q_\sigma}$ is $(0, \omega)$–elliptic

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- $$G(x_i, x_j) = -\omega_i \omega_j \left[\sum_{k=1}^{\min\{i,j\}-1} \frac{W_k^2}{c_k \omega_k \omega_{k+1}} + \sum_{k=\max\{i,j\}}^{n-1} \frac{(1 - W_k)^2}{c_k \omega_k \omega_{k+1}} - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1} \frac{W_k(1 - W_k)}{c_k \omega_k \omega_{k+1}} \right], \quad W_k = \sum_{j=1}^{k} \omega_j^2$$

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Consider $\mathcal{Q}$ the signless laplacian of the path on n vertices with conductances $c_1,\ldots,c_{n-1}>0$, $\sigma\in\Omega_+(V)$ and $\mathcal{Q}_{q_\sigma}$

- $$G(x_i,x_j)=(-)^{i+j}\sigma_i\sigma_j\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{W_k^2}{c_k\sigma_k\sigma_{k+1}}+\sum_{k=\max\{i,j\}}^{n-1}\frac{(1-W_k)^2}{c_k\sigma_k\sigma_{k+1}}-\sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{W_k(1-W_k)}{c_k\sigma_k\sigma_{k+1}}\right],\quad W_k=\sum_{j=1}^{k}\sigma_j^2$$

- $$R(x_i,x_j)=\sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{1}{c_k\sigma_k\sigma_{k+1}}$$

Example: A path

Consider $\mathcal{Q}$ the signless laplacian of the path on n vertices with conductances $c_1, \dots, c_{n-1} > 0$, $\sigma \in \Omega_+(V)$ and $\mathcal{Q}_{q_\sigma}$

- $$G(x_i,x_j) = (-)^{i+j}\sigma_i\sigma_j\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{W_k^2}{c_k\sigma_k\sigma_{k+1}} + \sum_{k=\max\{i,j\}}^{n-1}\frac{(1-W_k)^2}{c_k\sigma_k\sigma_{k+1}} - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{W_k(1-W_k)}{c_k\sigma_k\sigma_{k+1}}\right], \quad W_k = \sum_{j=1}^{k}\sigma_j^2$$

- $$R(x_i,x_j) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{1}{c_k\sigma_k\sigma_{k+1}} \Longrightarrow \mathsf{K} = \sum_{k=1}^{n-1}\frac{W_k(1-W_k)}{c_k\sigma_k\sigma_{k+1}}$$

Example: Single Perturbations on a Path

Consider $\mathcal{Q}$ the signless laplacian of the path on n vertices with conductances $c_1, \dots, c_{n-1} > 0$, $\sigma \in \Omega_+(V)$ and $\mathcal{Q}_{q_\sigma}$

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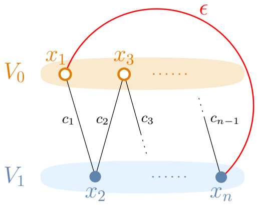

n even

$$Q^\epsilon(x_1)=(c_1+\epsilon)u(x_1)+c_1u(x_2)+\epsilon\,u(x_n)+q^\epsilon(x_1)u(x_1)$$

$$Q^\epsilon(x_j)=(c_j+c_{j-1})u(x_j)+c_ju(x_{j+1})+c_{j-1}u(x_{j-1})+q^\epsilon(x_j)u(x_j)$$

$$Q^\epsilon(x_n)=(c_{n-1}+\epsilon)u(x_n)+c_{n-1}u(x_{n-1})+\epsilon\,u(x_1)+q^\epsilon(x_n)u(x_n)$$

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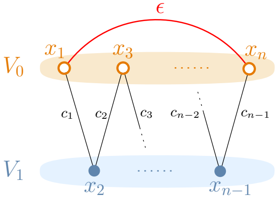

n odd

- $$Q^\epsilon(x_1)=(c_1+\epsilon)u(x_1)+c_1u(x_2)-\epsilon\,u(x_n)+q^\epsilon(x_1)u(x_1)$$
  $$Q^\epsilon(x_j)=(c_j+c_{j-1})u(x_j)+c_ju(x_{j+1})+c_{j-1}u(x_{j-1})+q^\epsilon(x_j)u(x_j)$$
  $$Q^\epsilon(x_n)=(c_{n-1}+\epsilon)u(x_n)+c_{n-1}u(x_{n-1})-\epsilon\,u(x_1)+q^\epsilon(x_n)u(x_n)$$

Example: Single Perturbations on a Path

Consider $\mathcal{Q}$ the signless laplacian of the path on n vertices with conductances $c_1, \dots, c_{n-1} > 0$, $\sigma \in \Omega_+(V)$ and $\mathcal{Q}_{q_\sigma}$

- $W_k = \sum_{j=1}^{k} \sigma_j^2, \;\; C = \dfrac{1}{\epsilon \sigma_1 \sigma_n} + \sum_{k=1}^{n-1} \dfrac{1}{c_k \sigma_k \sigma_{k+1}}, \;\; H = \sum_{k=1}^{n-1} \dfrac{W_k}{c_k \sigma_k \sigma_{k+1}}$

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- $$\begin{aligned} G^{\epsilon}(x_i, x_j) = (-)^{i+j} \sigma_i \sigma_j \Bigg[& \sum_{k=1}^{\min\{i,j\}-1} \frac{W_k^2}{c_k \sigma_k \sigma_{k+1}} + \sum_{k=\max\{i,j\}}^{n-1} \frac{(1-W_k)^2}{c_k \sigma_k \sigma_{k+1}} \\ & - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1} \frac{W_k(1-W_k)}{c_k \sigma_k \sigma_{k+1}} \Bigg] \\ & - \frac{(-1)^{i+j} \sigma_i \sigma_j}{C} \left[H - \sum_{k=i}^{n-1} \frac{1}{c_k \sigma_k \sigma_{k+1}} \right] \left[H - \sum_{k=j}^{n-1} \frac{1}{c_k \sigma_k \sigma_{k+1}} \right] \end{aligned}$$

Example: Single Perturbations on a Path

Consider $\mathcal{Q}$ the signless laplacian of the path on n vertices with conductances $c_1,\dots,c_{n-1}>0$, $\sigma\in\Omega_+(V)$ and $\mathcal{Q}_{q_\sigma}$

- $W_k=\displaystyle\sum_{j=1}^{k}\sigma_j^2,\ C=\frac{1}{\epsilon\sigma_1\sigma_n}+\sum_{k=1}^{n-1}\frac{1}{c_k\sigma_k\sigma_{k+1}},\ H=\sum_{k=1}^{n-1}\frac{W_k}{c_k\sigma_k\sigma_{k+1}}$

- $$R^\epsilon(x_i,x_j)=\sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{1}{c_k\sigma_k\sigma_{k+1}}-\frac{1}{C}\left[\sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{1}{c_k\sigma_k\sigma_{k+1}}\right]^2$$

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- $W_k = \displaystyle\sum_{j=1}^{k} \sigma_j^2, \quad C = \frac{1}{\epsilon \sigma_1 \sigma_n} + \sum_{k=1}^{n-1} \frac{1}{c_k \sigma_k \sigma_{k+1}}, \quad H = \sum_{k=1}^{n-1} \frac{W_k}{c_k \sigma_k \sigma_{k+1}}$

- $$\mathsf{K}^\epsilon = \sum_{k=1}^{n-1} \frac{W_k(1 - W_k)}{c_k \sigma_k \sigma_{k+1}} - \frac{1}{C} \sum_{j=1}^{n} \sigma_j^2 \left[H - \sum_{k=j}^{n-1} \frac{1}{c_k \sigma_k \sigma_{k+1}} \right]^2$$