Recovering piecewise constant conductances on networks

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Barcelona, 11/05/2023

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Network definition

A DC electrical network $\Gamma = (V, c)$ is an undirected weighted ***graph*** without loops nor multiple edges, with:

- A set of n ***vertices*** V.
- A ***conductance*** $c : V \times V \longrightarrow \mathbb{R}_{\geq 0}$, with $c_{jk} := c(j,k) = c(k,j)$ for all $j, k \in V$.
- A set of ***edges*** $E = \{(j,k) \in V \times V \text{ such that } c(j,k) > 0\}$.

Ohm's Law

The current in $(j,k) \in E$, I_{jk} is equal to its conductance multiplied by the difference between the potential in j, v_j and in k, v_k:

$$r_{jk} I_{jk} = (v_j - v_k).$$

Kirchhoff's Current Law

The current injected at each node j, I_j, is equal to the sum of currents leaving j:

$$I_j = \sum_{k:(j,k)\in E} I_{jk}.$$

Ohm's Law

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$$I_{jk} = c_{jk}(v_j - v_k) = \nabla v(j,k).$$

Kirchhoff's Current Law

The current injected at each node j, I_j, is equal to the sum of currents leaving j:

$$I_j = \sum_{k:(j,k)\in E} I_{jk} = -div(\nabla v)(j).$$

The ***Laplacian matrix*** L of a network Γ is the $n \times n$ real symmetric singular M-matrix defined by:

$$L(j,k) = \begin{cases} -c_{jk} & \text{if } (j,k) \in E \\ \sum_{t \neq j} c_{jt} & \text{if j=k} \\ 0 & \text{otherwise.} \end{cases}$$

The previous equations can be written as a linear map from the potential to the current, whose matrix is the Laplacian:

$$L \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} I_1 \\ \vdots \\ I_n \end{pmatrix}.$$

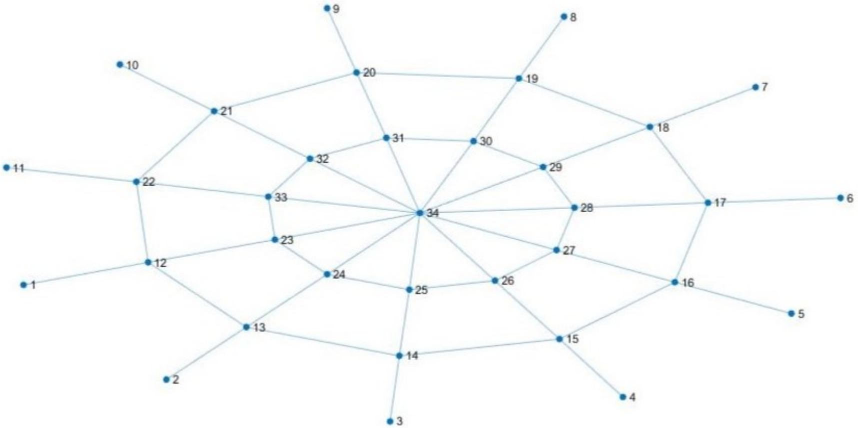
Networks with boundary

Definition

A ***network with boundary*** is a connected network in which there is a connected subset of interior nodes $F = \{m+1, ..., n\}$ with a totally disconnected boundary $\delta(F) = \{1, ..., m\}$ such that $V = \overline{F} = F \sqcup \delta(F)$.

Global equilibrium condition: It is not possible to inject current in the interior nodes, I_j=0 for all $j \in F$.

Example of a network with boundary: spider



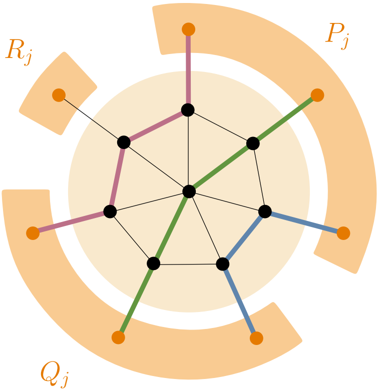
Spider network

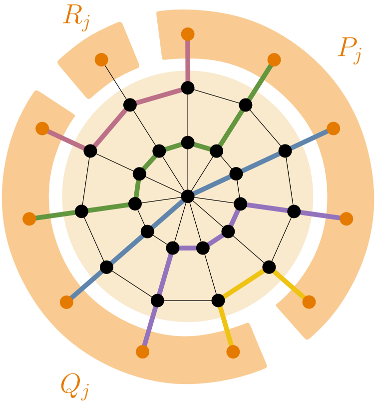
It has m boundary vertices of degree 1, $\delta(F) = \{1, \ldots, m\}$, which we sort in circular order.

The nodes in F have degree 4, but the central node, which has degree m.

We will work with spiders with $m \equiv 3 \pmod 4$ radii and $\dfrac{m-3}{4}$ circles, which are ***well–connected***.

Well-connected spider network





There are disjoint paths from each node in P_j to a node in Q_j.

Networks with boundary

Given a matrix M and A, B sets of indexes, $M(A;B)$ is obtained from M with rows indexed by A and columns indexed by B.

The voltage-current equations become in this case:

$$L\begin{pmatrix} v_{\delta(F)} \\ v_F \end{pmatrix} = \begin{pmatrix} D & -C(\delta(F);F) \\ -C(\delta(F);F)^T & L(F;F) \end{pmatrix}\begin{pmatrix} v_{\delta(F)} \\ v_F \end{pmatrix} = \begin{pmatrix} I_{\delta(F)} \\ 0 \end{pmatrix},$$

with D a diagonal matrix.

Dirichlet–to–Neumann matrix

The submatrix $L(F;F)$ is invertible, so under the global equilibrium condition, ***the potential in F is determined by the potential in*** $\delta(F)$:

$$v_F = L(F;F)^{-1} C(\delta(F);F)^T v_{\delta(F)}.$$

Furthermore, the relationship between voltage and current in the boundary nodes is linear, and the response matrix Λ is called the ***Dirichlet–to–Neumann matrix***. It is the Schur complement of $L(F;F)$ in L; that is:

$$\Lambda = L/L(F;F) = D - C(\delta(F);F)L(F;F)^{-1}C(\delta(F);F)^T,$$

$$\Lambda v_{\delta(F)} = I_{\delta(F)}.$$

Λ is a symmetric positive semidefinite M-matrix.

Inverse conductance problem

Inverse conductance problem

Given Λ***, estimate the conductance*** c of a network with boundary $\Gamma = (\overline{F}, c)$ with known set of nodes and set of edges.

Λ can be obtained from current and voltage measured data at the boundary $\delta(F)$. We obtain information about the interior of the network F from data at $\delta(F)$ (Discrete Electric impedance tomography).

It is the discrete analogous to the Calderón's inverse conductivity problem.

In the case of a well-connected spider network, the inverse conductance problem has unique solution, and an ***exact algorithm*** which calculates the conductances from Λ was created.

Both the continuous and discrete problem are ***exponentially ill-posed***: despite the algorithm is exact, any error in the entries of the Dirichlet-to-Neumann matrix (which are stored with finite precision) is amplified several orders of magnitude in the algorithm.

It is known that, in the continuous problem, if it is a-priori known that the conductivity is ***piecewise constant*** with a bounded number of unknown values, then the problem becomes Lipschitz stable.

Idea: look for a discrete analogous to being piecewise constant.

We assume that we know a-priori that there are is a ***partition*** of the set of edges $E = E_1 \sqcup ... \sqcup E_s$, $s \ll |E|$ such that the conductance is ***constant*** at each subset.

It is not known how to enforce the piecewise constant hypothesis in the exact algorithm.

If we formulate the problem as a ***polynomial optimization problem***, we can add a term of deviation respect to being piecewise constant in the partition as a penalty to the objective function.

Notation

Let $\Gamma = (\bar{F}, c)$ be a well-connected spider network with Laplacian matrix L and Dirichlet-to-Neumann matrix Λ. We denote by e^t the t-th vector of the standard basis in $\mathbb{R}^m$. Let v_F^t be the unique vector such that:

$$L \begin{pmatrix} e^t \\ v_F^t \end{pmatrix} = \begin{pmatrix} \Lambda(\delta(F); \{t\}) \\ 0 \end{pmatrix}. \qquad (1)$$

We denote by v_j^t the j-th component of $v^t := \begin{pmatrix} e^t \\ v_F^t \end{pmatrix} \in \mathbb{R}^n$.

Problem 1: Conductance recovery

Given a well-connected spider network, a Dirichlet-to-Neumann matrix Λ, a partition $E = E_1 \sqcup ... \sqcup E_s$ and a penalty parameter $\lambda \geq 0$; determine a value of the variables:

$c_{jk} \geq 0$ for all $(j,k) \in E$;

c_j for all $j = 1, .., s$; and

v_j^t for all $j = m+1, ..., n$ and $t = 1, ..., m$ which minimize:

$$p = \sum_{t=1}^{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{n} c_{jk} \left(v_j^t - v_k^t \right) - \Lambda \left(\{j\}; \{t\} \right) \right)^2 + \lambda \sum_{i=1}^{s} \sum_{(j,k) \in E_i} \left(c_{jk} - c_i \right)^2$$

such that, for all $j = m+1, ..., n$ and $t = 1, ..., m$:

$$g_j^t := \sum_{k=1}^{n} c_{jk} \left(v_j^t - v_k^t \right) = 0$$

Problem 1: Conductance recovery

Given a well-connected spider network, a Dirichlet-to-Neumann matrix Λ, a partition $E = E_1 \sqcup ... \sqcup E_s$ and a penalty parameter $\lambda \geq 0$; determine a value of the variables:
$c_{jk} \geq 0$ for all $(j, k) \in E$;
c_j for all $j = 1, .., s$; and
v_j^t for all $j = m+1, ..., n$ and $t = 1, ..., m$ which minimize:

$$p = \|\Lambda' - \Lambda\|_F^2 + \lambda \|c - c_{cte}\|_2^2,$$

such that, for all $j = m+1, ..., n$ and $t = 1, ..., m$:

$$g_j^t := \sum_{k=1}^{n} c_{jk} \left(v_j^t - v_k^t\right) = 0 \qquad (2)$$

From (1), the (j,t)-th entry of the Dirichlet-to-Neumann matrix Λ' of the network with the conductances c_{jk} we recover is equal to $\sum_{k=1}^{n} c_{jk}\left(v_j^t - v_k^t\right)$, where $(v_1^t, ..., v_m^t)^T = e^t$.

The voltage variables v_j^t with $j = m+1, ..., n$ must satisfy equations (2), which are the condition of having no injected current in the interior F.

For each $j = 1, .., s$, the variable c_j is a unknown conductance. We denote by c_{cte} the piecewise constant conductance on the partition $E = E_1 \sqcup ... \sqcup E_s$ which is equal to c_j on each E_j.

We minimize the deviation in the Dirichlet-to-Neumann matrix plus the parameter λ multiplied by a term ***penalizing*** the deviation in the conductance respect to being piecewise constant in the partition.

The case $\lambda \to \infty$ corresponds with enforcing the hypotesis that the recovered conductance is piecewise constant on $E = E_1 \sqcup \ldots \sqcup E_s$ and minimizing the difference between Λ' and Λ. The case $\lambda = 0$ corresponds with minimizing the difference between Λ' and Λ ignoring the piecewise constant hypotesis.

We denote by:

- r the number of variables of the problem.
- $J =< g_j^t$ such that $j = m+1, ..., n;\ \ t = 1, ..., m >$ the ideal generated by the quadratic polynomials in (2).
- $V(J) \subset \mathbb{R}^r$ the real vanishing set of J.
- $I(V(J))$ the ideal of polynomials vanishing on $V(J)$.

Then, the optimization problem can be stated as finding a minimum of the quartic p in $\left(\mathbb{R}^{|E|}_{\geq 0} \times \mathbb{R}^{r-|E|}\right) \cap V(J)$.

Optimality guarantees of the recovered conductances

Using a numerical optimization method such an interior-point method, we can obtain a ***minimum*** of p, $y^* \in \left(\mathbb{R}^{|E|}_{\geq 0} \times \mathbb{R}^{r-|E|}\right) \cap V(J)$ with $p(y^*) = z$, but a-priori ***there is no guarantee that it is a global minimum*** and thus a solution to Problem 1.

In the case $z = 0$, we know y^* is a global minimum, and it is unique, because the Dirichlet-to-Neumann matrix of the network with the recovered conductances is Λ, and if $\lambda > 0$, then the piecewise constant conductance hypotesis is true.

Optimality guarantees of the recovered conductances

In a general case, we can try to find a guarantee that y^* is a global minimum using techniques of ***Sum of Squares (SOS)*** decompositions, which are relaxations used in polynomial optimization to obtain a lower bound for a real polynomial in a real algebraic variety.

We formulate the following ***SOS*** problem:

Problem 2

Given a bound $d \in \mathbb{N}$, the quartic p and the variety $V(J)$ previously defined, and a value $z \geq 0$, is there any polynomial h such that: $p(y) - z = h(y)$ in $\mathbb{R}[y]/I(V(J))$; h is SOS, and $deg(h) \leq 2d$?

Sum of squares (SOS)

If $y^* \in \left(\mathbb{R}_{\geq 0}^{|E|} \times \mathbb{R}^{r-|E|}\right) \cap V(J)$ with $p(y^*) = z$, and Problem 2 has an affirmative answer for p, $V(J)$, z and any d, then $p(y^*) \geq z$ in $V(J)$, so y^* is a solution to Problem 1.

There are nonnegative polynomials which are not sum of squares (Hilbert's 17th problem), but in some cases nonnegativity and being SOS are equivalent, and in other cases, we can still find a good lower bound z for the global minimum.

Sum of squares (SOS)

Given a Gröbner basis of $I(V(J))$, Problem 2 reduces to a ***semidefinite program*** (SDP).

It is equivalent to find a symmetric positive semidefinite matrix Q such that the normal form of $p - z - u^T Qu$ in the Gröbner basis is zero, where u is a vector whose entries are the standard monomials corresponding to the Gröbner basis, (that is, the monomials which are not divisible by any leading term of the polynomials in that basis), of degree $\leq d$.

Real Hilbert's Nullstellensatz

Theorem (*Real Hilbert's Nullstellensatz*)

For any ideal J, $I(V(J)) = \sqrt[\mathbb{R}]{J}$, the ***real radical*** of J.

$$\sqrt[\mathbb{R}]{J} = \left\{ f \in \mathbb{R}[x_1, ..., x_n] | -f^{2k} = \sum_i h_i^2 + g \ \ for \ \ some \right.$$
$$\left. k \in \mathbb{N}_{\geq 0}, h_i \in \mathbb{R}[x_1, ..., x_n] \ \ and \ \ g \in J \right\}$$

In general, it is ***computationally complex*** to compute a Gröbner basis of the real radical $I(V(J))$ of J. Alternatively, we can check if $p - z$ is sum of squares in $\mathbb{R}[y]/J$, which is a SDP problem that only requires a Gröbner basis of J, and obtaining an affirmative answer to this problem is a sufficient condition for obtaining an affirmative answer to Problem 2.

Initial guess for the interior point algorithm

For every conductance variable we choose as an ***initial guess*** the conductance of the ***spider with constant conductance*** c whose Dirichlet-to-Neumann matrix is closest to Λ.

Lemma

Given a network with boundary and its Dirichlet-to-Neumann matrix Λ, then, the network with the same topology and constant conductance c at all edges whose Dirichlet-to-Neumann matrix is closest to Λ in the Frobenius norm satisfies that c is the solution to the following Non-Negative Least Squares (NNLS) problem:

$$min_c ||\Lambda - c(I - L'(\delta(F);F)L'(F;F)^{-1}L'(\delta(F);F)^T)||_F^2;$$

where L' is the (unweighted) combinatorial laplacian of the graphs.

In the case of a well connected spider, if $m \geq 7$, the NNLS problem in the previous lemma becomes:

$$min_c ||\Lambda - c(I - G'(\{1,..,m\}\,;\{1,..,m\}))||_F^2,$$

where $G' = L'(F;F)^{-1}$; and if $m = 3$, the NNLS problem becomes:

$$min_c \left\| \Lambda - \frac{c}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \right\|_F^2 .$$

Initial guess for voltage variables

We define L as the Laplacian matrix of the spider network with every conductance equal to the initial guess, and we choose as an ***initial guess*** for every voltage variable v_j^t, the voltage in that spider in the j-th node which gives ***zero current in F*** when the voltage in $\delta(F)$ is equal to e^t.

$$v_F = L(F;F)^{-1} C(\delta(F);F)^T e^t.$$

Experimental results

We use an interior point algorithm with tolerance equal to 10^{-8} to recover the conductances.

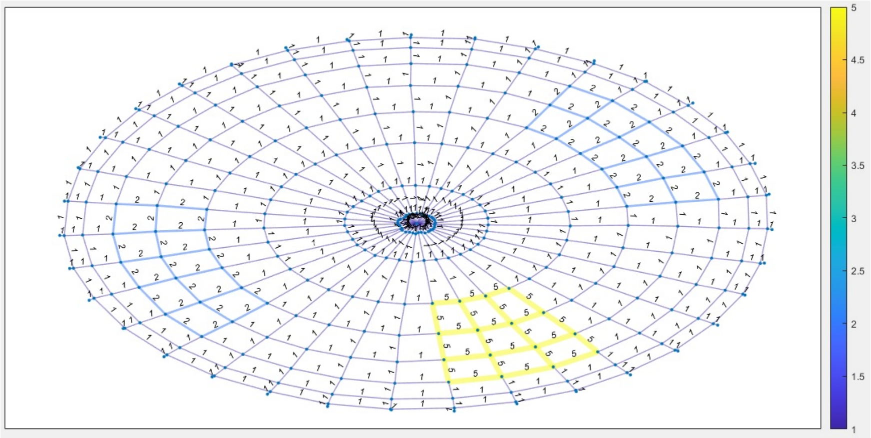
We measure the error as the Frobenius norm of the difference between the Laplacian L of the real spider and the Laplacian L' of the recovered one.

We also use the maximum of the absolute value of the difference between the conductance c of the real spider and the conductance c' of the recovered one.

Example 1: Comparison with the exact algorithm of conductance recovery

We recover several spider networks with piecewise constant conductance, using the exact algorithm and also using our approach introducing the true hypothesis as a penalty with a value $\lambda > 0$. We compare the error in both approaches for different values of m and s.

The ***numerical results improve drastically*** in large networks due to the penalty.


5
4.5
4
3.5
3
2.5
2
1.5
1

$||L - L'||_F$ with our approach (with penalty):

$m \backslash s$	1	3	5	8	10	40
11	$5 \cdot 10^{-9}$	$6 \cdot 10^{-10}$	$3 \cdot 10^{-9}$	10^{-7}	$3 \cdot 10^{-8}$	$9 \cdot 10^{-5}$
15	$6 \cdot 10^{-9}$	10^{-9}	10^{-8}	10^{-9}	10^{-7}	$3 \cdot 10^{-5}$
19	$5 \cdot 10^{-11}$	$6 \cdot 10^{-10}$	10^{-7}	$6 \cdot 10^{-9}$	10^{-8}	$3 \cdot 10^{-4}$
35	$4 \cdot 10^{-8}$	$6 \cdot 10^{-9}$	$2 \cdot 10^{-8}$	10^{-8}	$2 \cdot 10^{-8}$	10^{-6}

$||L - L'||_F$ with the exact algorithm:

$m \backslash s$	1	3	5	8	10	40
11	$2 \cdot 10^{-9}$	$2 \cdot 10^{-9}$	$2 \cdot 10^{-4}$	$5 \cdot 10^{-7}$	10^{-6}	$3 \cdot 10^{-6}$
15	$1 \cdot 10^{-5}$	$7 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-1}$	$4 \cdot 10^{-1}$
19	$4 \cdot 10^{-1}$	$7 \cdot 10^{-1}$	9	$4 \cdot 10^{4}$	$7 \cdot 10^{3}$	$6 \cdot 10^{3}$
35	$4 \cdot 10^{4}$	$7 \cdot 10^{5}$	$4 \cdot 10^{4}$	$2 \cdot 10^{4}$	$3 \cdot 10^{4}$	$6 \cdot 10^{4}$

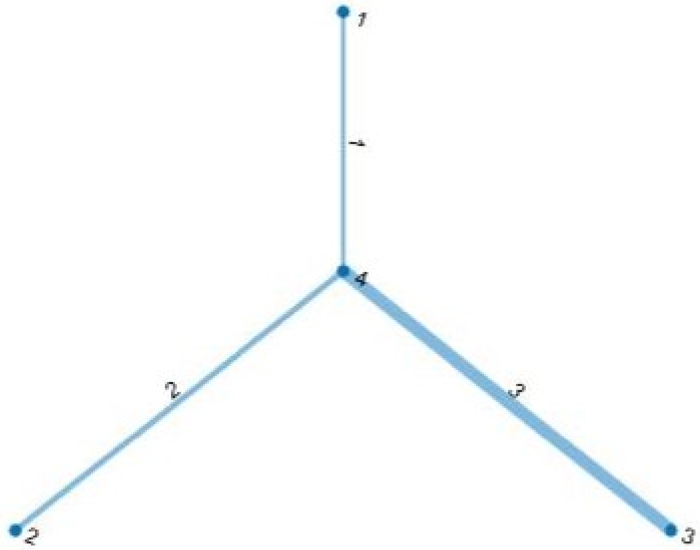
Example 2: Optimality guarantees when the piecewise constant conductance hypotesis is not true

Spider network with $m = 3$ radii, and $c_{14} = 1$, $c_{24} = 2$, $c_{34} = 3$.

$$L = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & -3 \\ -1 & -2 & -3 & 6 \end{pmatrix}$$

$$\Lambda = \frac{1}{6} \begin{pmatrix} 5 & -2 & -3 \\ -2 & 8 & -6 \\ -3 & -6 & 9 \end{pmatrix}$$

Well-connected spider network with m=3



We recover the conductances of the spider under the hypothesis that the conductance is constant on the partition $E = E_1 \sqcup E_2$, with $E_1 = \{(1,4),(2,4)\}$, $E_2 = \{(3,4)\}$.

We set $\lambda = 1$ as the penalty parameter.

The hypotesis is not true, so there is not any conductance which has zero error.

We do not need a conductance variable c_2 because E_2 has only one edge. We denote $c := c_1$.

Find a value of $c_{14} \geq 0, c_{24} \geq 0, c_{34} \geq 0, c, v_4^1, v_4^2, v_4^3$ minimizing

$$p = \left(c_{14} - c_{14} v_4^1 - \frac{5}{6}\right)^2 + \left(-c_{24} v_4^1 + \frac{1}{3}\right)^2 + \left(-c_{34} v_4^1 + \frac{1}{2}\right)^2 +$$

$$+ \left(-c_{14} v_4^2 + \frac{1}{3}\right)^2 + \left(c_{24} - c_{24} v_4^2 - \frac{4}{3}\right)^2 + \left(-c_{34} v_4^2 + 1\right)^2 +$$

$$+ \left(-c_{14} v_4^3 + \frac{1}{2}\right)^2 + \left(-c_{24} v_4^3 + 1\right)^2 + \left(c_{34} - c_{34} v_4^3 - \frac{3}{2}\right)^2 +$$

$$+ (c_{14} - c)^2 + (c_{24} - c)^2 .$$

subject to:

$$\begin{cases} g_4^1 := -c_{14} + (c_{14} + c_{24} + c_{34}) v_4^1 = 0 \\ g_4^2 := -c_{24} + (c_{14} + c_{24} + c_{34}) v_4^2 = 0 \\ g_4^3 := -c_{34} + (c_{14} + c_{24} + c_{34}) v_4^3 = 0 \end{cases}$$

We define $J = < g_4^1, g_4^2, g_4^3 >$

A Gröbner basis of J is $J = < h_1, h_2, h_3, h_4, h_5, h_6 >$

$$\begin{cases} h_1 = c_{24} - c_{14} + c_{34} + c_{14} v_4^1 - c_{24} v_4^2 - 2c_{24} v_4^3 - c_{34} v_4^3 \\ h_2 = c_{24} v_4^1 - c_{24} + c_{24} v_4^2 + c_{24} v_4^3 \\ h_3 = c_{34} v_4^1 - c_{34} + c_{24} v_4^3 + c_{34} v_4^3 \\ h_4 = c_{14} v_4^2 - c_{24} + c_{24} v_4^2 + c_{24} v_4^3 \\ h_5 = c_{34} v_4^2 - c_{24} v_4^3 \\ h_6 = c_{14} v_4^3 - c_{34} + c_{24} v_4^3 + c_{34} v_4^3 \end{cases}$$

With an interior point algorithm, we obtain a minimum of p, $y^* = (c_{14}, c_{24}, c_{34}, c, v_4^1, v_4^2, v_4^3)^T$, with $c_{14} = 1{,}1486$, $c_{24} = 1{,}5765$, $c_{34} = 3{,}4764$, $c = 1{,}3625$, $v_4^1 = 0{,}1852$, $v_4^2 = 0{,}2542$, $v_4^3 = 0{,}5606$.

$p(y^*) = 0{,}1995$.

We check with a SDP solver there is a ***SOS*** polynomial equal to $p - 0{,}1995$ in $\mathbb{R}[y]/J$, so $p \geq 0{,}1995$ in $V(J)$, and y^* is a solution to Problem 1.

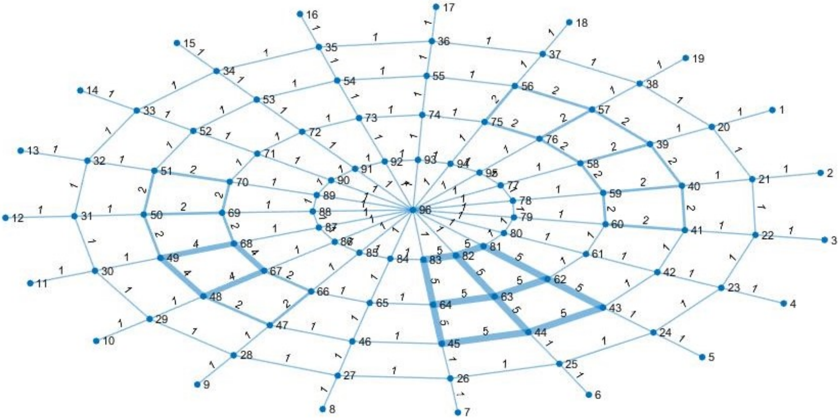
We do not need a Gröbner basis of $I(V(J))$ in this case.

Example 3: Variation of the error in the recovered conductances with the penalty parameter λ

In the tested cases, if we assume a-priori a piecewise constant conductance ***hypothesis which holds*** in the real network, there is (almost) ***zero error*** with any value $\lambda > 0$. If we set $\lambda = 0$, there is error due to the ill-posedness of the problem.

We show how the error in recovering the conductances of a spider with $m = 19$ varies with λ when a piecewise constant conductance ***hypothesis which does not hold*** in the network is used.

Well-connected spider network with $m = 19$



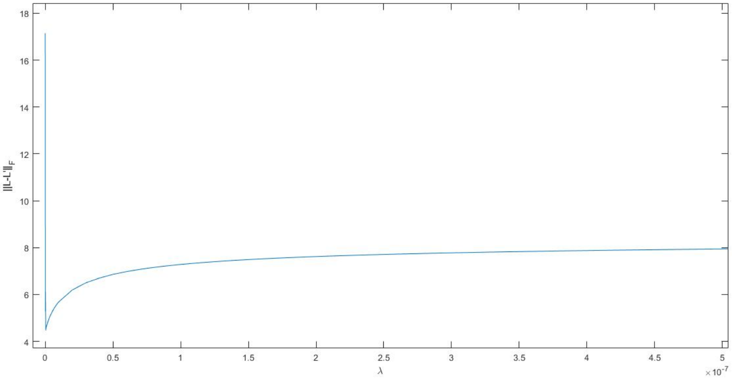
The conductance is piecewise constant on a partition $E = E_1 \sqcup ... \sqcup E_4$, with:

- $c(E_1) = 1$,
- $c(E_2) = 5$,
- $c(E_3) = 2$,
- $c(E_4) = 4$.

We suppose that we know a-priori the ***false hypothesis*** that the conductance is piecewise constant in a partition $E = A_1 \sqcup ... \sqcup A_3$, with:

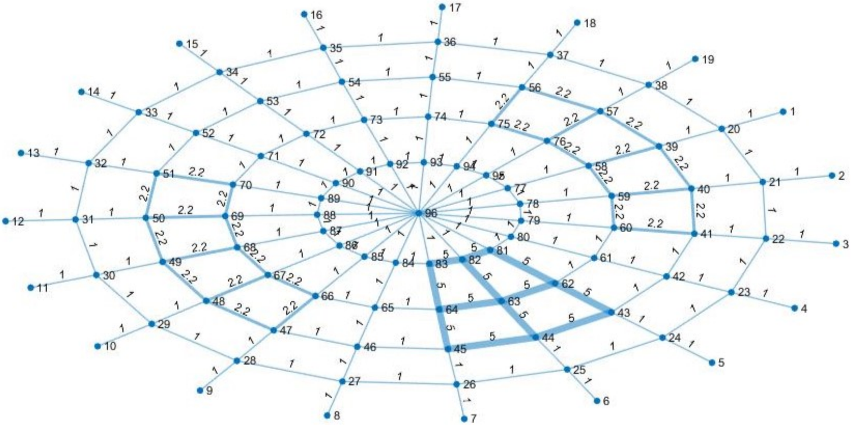
- $A_1 = E_1$,
- $A_2 = E_2$,
- $A_3 = E_3 \sqcup E_4$,

Error in the recovered conductances as a function of λ

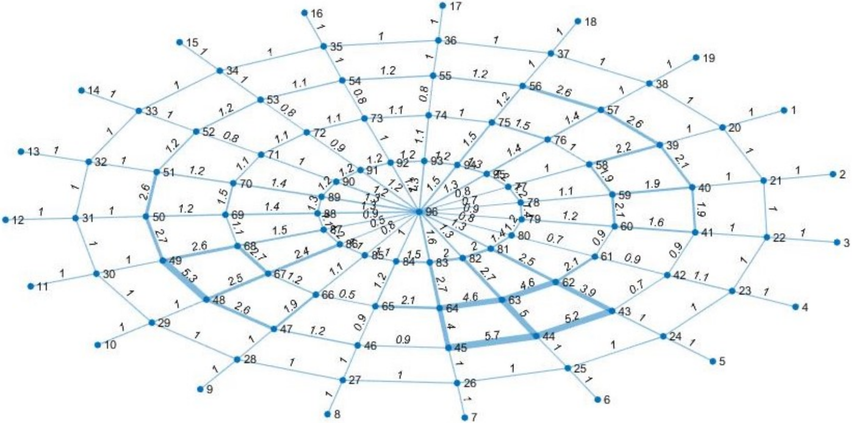


λ	$\|\|L-L'\|\|_F$	$max\|c-c'\|$	$\|\|\Lambda-\Lambda'\|\|_F$
0	17,1476	2,9997	$5{,}9512\cdot 10^{-6}$
$4\cdot 10^{-10}$	4,4675	1,3334	$1{,}0791\cdot 10^{-5}$
10^5 $(\lambda\to\infty)$	8,8543	1,8057	0,0078

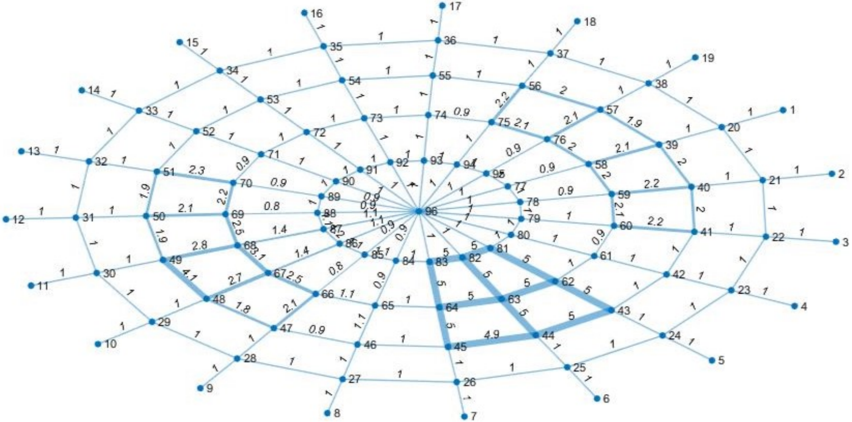
Results of recovering the conductances with $\lambda = 10000$



Results of recovering the conductances with $\lambda = 0$



Results of recovering the conductances with $\lambda = 4 \cdot 10^{-10}$



Even if we choose a piecewise constant hypotesis which is not true, in some cases the error in the recovered conductances due to imposing that hypotesis (case $\lambda \longrightarrow \infty$) is lower than the error due to the fact that the problem is ill-posed if we do not use any hypotesis (case $\lambda = 0$).

Introducing the ***piecewise constant*** hypotesis as a ***penalty*** rather than imposing it lets us consider intermediate values of λ, which can lead to good solutions, in which we have the flexibility to deviate respect to the hypotesis and we avoid the instabilities of the recovery without hypotesis ($\lambda = 0$).

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