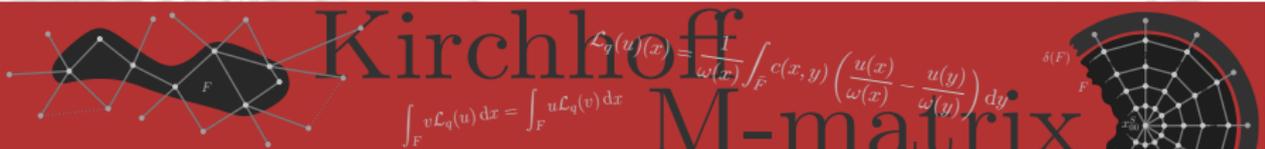




VI JORNADAS ALAMA: Laplacian and M-matrices on Graphs



M-matrices on graphs and their applications

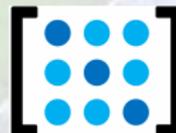
A glimpse on discrete vector calculus and potential theory

Andrés M. Encinas



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BARCELONATECH

Department of Mathematics



RED ALAMA

VI JORNADAS ALAMA: Laplacian and M-matrices on Graphs

Outline

- 1 Mimetic Discretizations of Continuum Mechanics:
Discrete vector calculus
- 2 The setting: **Graphs and networks**
- 3 Discrete Potential Theory. **Why here?**
- 4 Where is the Laplacian in the discrete setting?
- 5 Are there M -matrices around here? **Schrödinger operators**
- 6 Why we call Laplacian (or Schrödinger) the main operators on graphs?
- 7 Discrete Inverse Problems: **Dirichlet-to-Neumann maps**

Introducing Mimetic Methods

- ▶ **Boundary Value Problems** appears as the **state equations** for **mathematical physics** or **engineering problems**.
 - ↪ Their effective solution require **some kind of approximation**.
- ▶ **Mimetic Methods** model this problems directly on a **discrete space** **instead of discretizing the equations**
- ▶ **Mimetic Methods** are based on **vector calculus** on **weighted graphs**.
 - ↪ Its main ingredients are **functions** and **vector fields**, related each other through **discrete difference operators**.
- ▶ This formulation **does not requires highly regular structures**. It can be applied to represent **abstract data** with **irregular interrelationships**.
- ▶ Ideas come from **Algebraic Topology** or **Differential Geometry**.

Calculus on finite weighted graphs

From [Wikipedia](#), the free encyclopedia

In mathematics, **calculus on finite weighted graphs** is a discrete calculus for functions whose domain is the vertex set of a **graph** with a **finite** number of **vertices** and **weights** associated to the **edges**.

Calculus on finite weighted graphs

From [Wikipedia](#), the free encyclopedia

calculus on finite weighted graphs (networks)

graph **finite** **vertices** **weights**
edges. This involves formulating discrete operators on graphs which are analogous to differential operators in calculus, such as graph Laplacians (or **discrete Laplace operators**) as **discrete versions of the Laplacian**, and using these operators to formulate differential equations, difference equations, or variational models on graphs

Calculus on finite weighted graphs

From [Wikipedia](#), the free encyclopedia

calculus on finite weighted graphs (networks)

graph

finite

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discrete Laplace operators

discrete versions of the Laplacian

which can be interpreted as discrete versions of partial differential equations or continuum variational models.

Calculus on finite weighted graphs

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calculus on finite weighted graphs (networks)

graph

finite
edges

vertices

weights

discrete Laplace operators

discrete versions of the Laplacian

interpreted as discrete versions of partial differential equations or continuum variational models. Such equations and models are important tools to mathematically model, analyze, and process discrete information in many different research fields, e.g., [image processing](#), [machine learning](#), and [network analysis](#).

Calculus on finite weighted graphs

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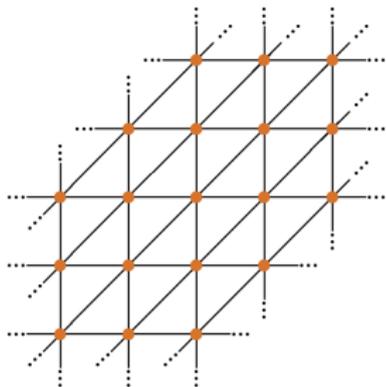
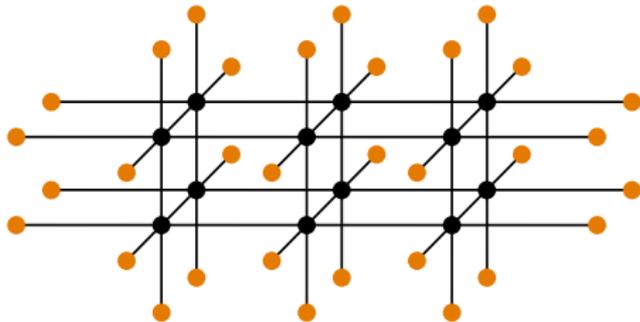
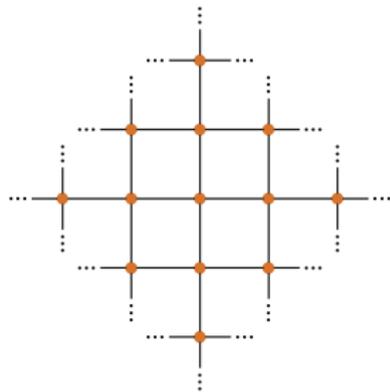
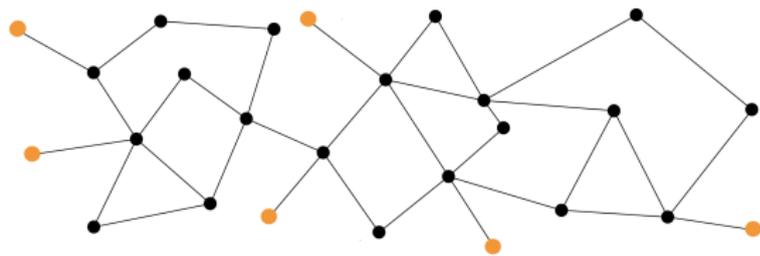
image processing

machine learning

network analysis

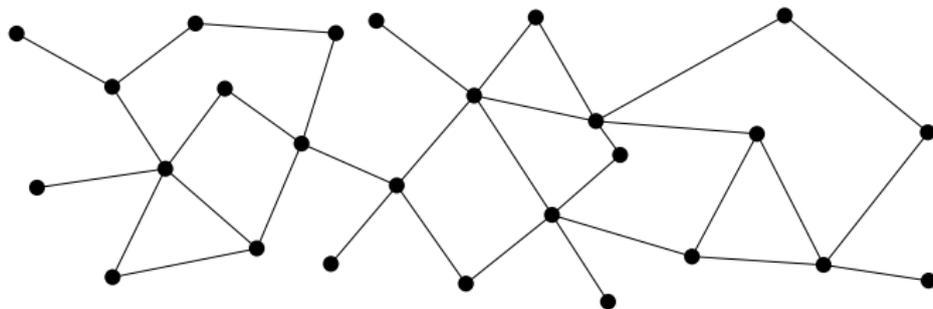
↪ See more references on [Calculus_on_finite_weighted_graphs](#)

Graphs



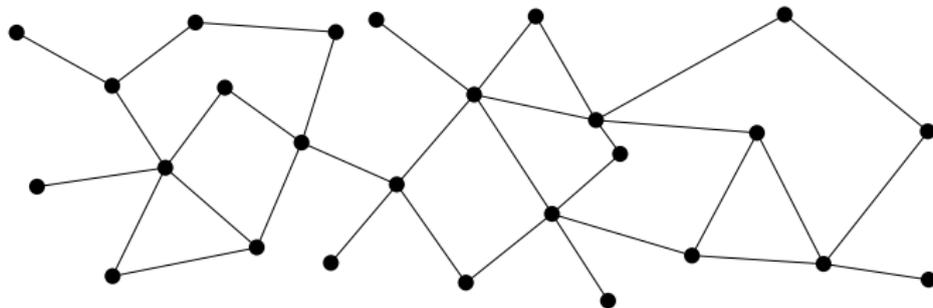
What is a finite Network?

► Graph $\Gamma = (V, E)$: V vertices, E edges (both **finite**)



What is a finite Network?

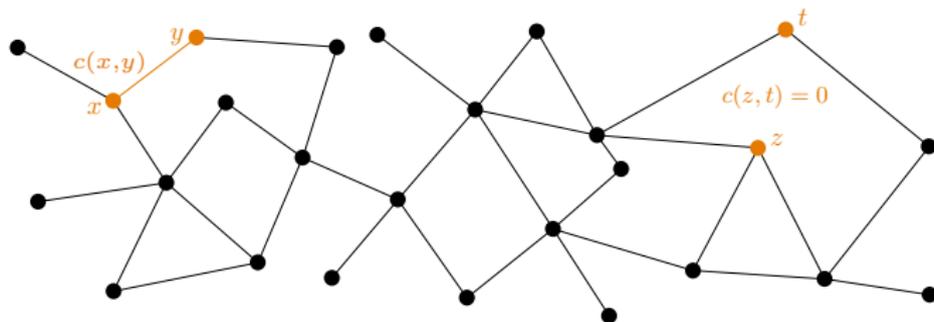
- ▶ Graph $\Gamma = (V, E)$: V vertices, E edges (both **finite**)



- ▶ Adjacency: $x \sim y \Leftrightarrow \{x, y\} = e_{xy} \in E$
 $\rightsquigarrow V(x) = \{y : y \sim x\}$

What is a finite Network?

► Graph $\Gamma = (V, E)$: V vertices, E edges



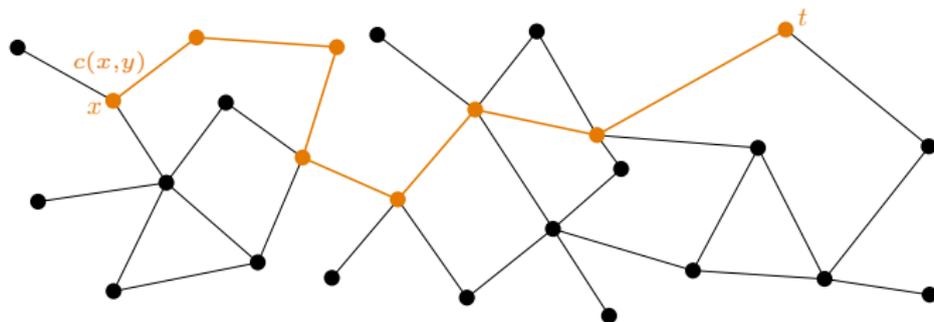
► Conductance: $c: V \times V \rightarrow [0, +\infty)$ (Sometimes called weight)

- ① $c(x, x) = 0, x \in V$
- ② $c(x, y) = c(y, x), x, y \in V$ (symmetry)
- ③ $c(x, y) > 0$ iff $\{x, y\} = e_{xy} \in E$

► (Generalized) Degree: $\kappa(x) = \sum_{y \sim x} c(x, y) = \sum_{y \in V} c(x, y)$

What is a finite Network?

- ▶ Connected Network: $\Gamma = (V, c): E = \{\{x, y\} \in V \times V : c(x, y) > 0\}$



- ▶ Conductance: $c: V \times V \rightarrow [0, +\infty)$ symmetric and $c(x, x) = 0$

- ▶ (Generalized) Degree: $\kappa(x) = \sum_{y \sim x} c(x, y) = \sum_{y \in V} c(x, y)$

- ▶ We will always assume that Γ is connected

Functions on graphs

Connected (finite) graph $\Gamma = (V, E)$: V vertices, E edges

- ▶ Space of functions: $\mathcal{C}(V) = \{u: V \rightarrow \mathbb{R}\}$ and $\mathcal{C}(E) = \{f: E \rightarrow \mathbb{R}\}$
- ▶ Support of $u \in \mathcal{C}(V)$: $\text{supp}(u) = \{x \in V : u(x) \neq 0\}$
- ▶ Space of functions supported by $F \subset V$: $\mathcal{C}(F) \equiv \{u: F \rightarrow \mathbb{R}\}$
- ▶ Weights: $\Omega(V) = \{\omega \in \mathcal{C}(V) : \omega(x) > 0, x \in V\}$
- ▶ Integral: Given $\nu \in \Omega(V)$, $\int_V u d\nu = \int_V u \nu dx = \sum_{x \in V} u(x) \nu(x)$
- ▶ Inner products in $\mathcal{C}(V)$: $\nu \in \Omega(V)$, $\int_V uv d\nu = \sum_{x \in V} u(x)v(x)\nu(x)$

Linear Operators on a graph

- ▶ Linear Operator: $\mathcal{K}: \mathcal{C}(V) \longrightarrow \mathcal{C}(V) \Rightarrow \mathcal{K}(x, y) = \mathcal{K}(\varepsilon_y)(x)$
- ▶ Kernel: $K \in \mathcal{C}(V \times V) \Rightarrow \mathcal{K}(u)(x) = \int_V K(x, y)u(y)dy, u \in \mathcal{C}(V)$
- ▶ If $V = \{x_1, \dots, x_n\}$: functions \simeq vectors and kernels \simeq matrices

$$\mathbf{K} = \begin{bmatrix} K(x_1, x_1) & \cdots & K(x_1, x_n) \\ \vdots & \ddots & \vdots \\ K(x_n, x_1) & \cdots & K(x_n, x_n) \end{bmatrix} \Leftrightarrow \mathcal{K}(u) = \mathbf{K}u$$

- ▶ \mathcal{K} is self-adjoint iff $\mathcal{E}(u, v) = \int_V v\mathcal{K}(u)dx = \int_V u\mathcal{K}(v)dx = \mathcal{E}(v, u)$

Linear Operators on a graph

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▶ \mathcal{K} self-adjoint $\Leftrightarrow K$ is symmetric $\Leftrightarrow \mathbf{K}$ is symmetric

▶ \mathcal{K} positive semidefinite $\Leftrightarrow \mathcal{E}(u, u) \geq 0 \Leftrightarrow \mathbf{K}$ is positive semidefinite

Linear Operators on a graph

▶ Linear Operator: $\mathcal{K}: \mathcal{C}(V) \longrightarrow \mathcal{C}(V) \Rightarrow \mathcal{K}(x, y) = \mathcal{K}(\varepsilon_y)(x)$

▶ Kernel: $K \in \mathcal{C}(V \times V) \Rightarrow \mathcal{K}(u)(x) = \int_V K(x, y)u(y)dy, u \in \mathcal{C}(V)$

▶ Other spaces of functions (taking adjacency into account)

$$\mathcal{C}(\Gamma) = \{f: V \times V \longrightarrow \mathbb{R} : f(x, y) = 0, \text{ if } x \not\sim y\}$$

$$\mathcal{C}(\Gamma \times \Gamma) = \{m: V \times V \times V \longrightarrow \mathbb{R} : m(x, y, z) = 0, \text{ if } x \not\sim y \text{ or } x \not\sim z\}$$

⋮

What is Potential Theory?

- **Potential Theory:** Study of **harmonic functions**; that is, functions that satisfy the **Laplace equation**

$$\frac{\partial u}{\partial x_1^2} + \cdots + \frac{\partial u}{\partial x_n^2} = 0$$

What is Potential Theory?

- **Potential Theory:** Study of harmonic functions; that is, functions that satisfy the Laplace equation

$$\frac{\partial u}{\partial x_1^2} + \cdots + \frac{\partial u}{\partial x_n^2} = 0$$

or related (elliptic) (Schrödinger) equations

$$-\left(\frac{\partial u}{\partial x_1^2} + \cdots + \frac{\partial u}{\partial x_n^2}\right) = f$$

$$-\left(\frac{\partial u}{\partial x_1^2} + \cdots + \frac{\partial u}{\partial x_n^2}\right) + qu = f$$

$$-\underbrace{\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\underbrace{k_{ij} \sum_{j=1}^n \frac{\partial u}{\partial x_j}}_{\nabla} \right)}_{\text{div}} + qu = f$$

What is Potential Theory?

- ▶ **Potential Theory:** Study of **harmonic functions**; that is, functions that satisfy the **Laplace equation**
- ▶ **19th-century physics** model of the **forces of nature**

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- ▶ **Modern Potential Theory:** Is defined in **axiomatic terms**

What is Potential Theory?

- ▶ **Potential Theory:** Study of harmonic functions; that is, functions that satisfy the Laplace equation
- ▶ 19th-century physics model of the forces of nature
- ▶ **Modern Potential Theory:** Is defined in axiomatic terms
- ▶ **The Holy Trinity of Modern Potential Theory**



M. Brelot 1903-1987



G. Choquet 1915-2006



J. Deny 1916-2016



Séminaire BCD 1957-1972

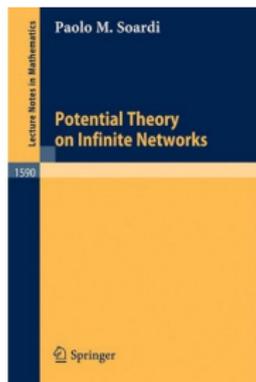
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- ▶ **Modern Potential Theory:** It is defined in **axiomatic terms**
 - **Linear aspect** \rightsquigarrow Analysis of **Kernels**
 - **Quadratic aspect** \rightsquigarrow Analysis of **(semi)-norms**
 - **Probability** \rightsquigarrow Study of **Markov chains**

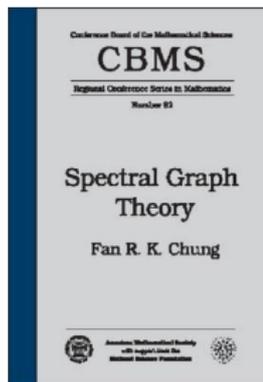
What is Potential Theory in this setting?

- ▶ **Locally finite state space case** \rightsquigarrow **Electrical network**
- ▶ **Finite state space case** \rightsquigarrow **Finite Electrical network**
 - ① Potentials, identified with **vectors**
 - ② Kernels, identified with **matrices**
 - ③ Resistance inversely proportional to **transition probabilities**

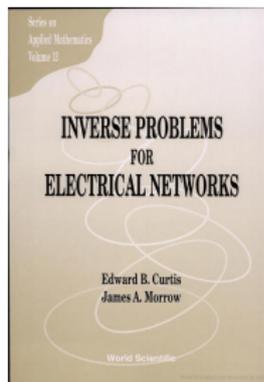
Potential Theory on Networks: Some references



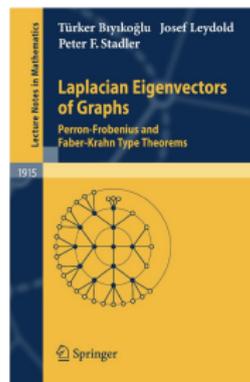
1994



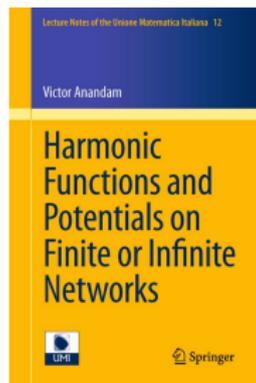
1997



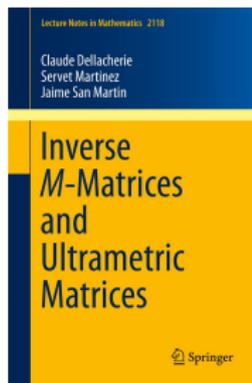
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2007



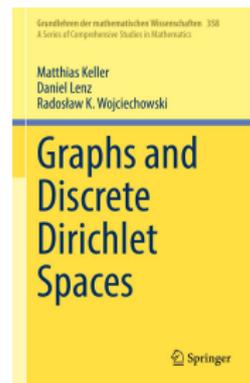
2011



2014



2018



2021

Journal d'Analyse Mathématique 5 (1956), 77-135

MODÈLES FINIS EN THÉORIE DU POTENTIEL *

Par

Gustave Choquet et Jacques Deny
à Paris à Strasbourg, France

Introduction

La théorie moderne du potentiel est l'étude de certains noyaux définis sur un espace localement compact X , et possédant des propriétés analogues à celles du noyau newtonien dans l'espace euclidien. La première difficulté qu'on rencontre dans cet essai de généralisation est précisément la définition du mot "noyau". Une idée naturelle est d'appeler ainsi toute fonction $G(x, y)$ définie sur l'espace produit $X \times X$, à valeurs réelles ≥ 0 , finies ou infinies; le potentiel $G\mu$ engendré par la mesure de Radon $\mu \geq 0$ sera la fonction

$$G\mu(x) = \int G(x, y) d\mu(y)$$

qui est définie pour tout $x \in X$ sous des hypothèses assez larges, telles que la semi-continuité de G , qu'on admet généralement. On s'intéresse à diverses propriétés pouvant être vérifiées par le noyau et concernant les potentiels. Parmi ces propriétés, signalons les trois suivantes, que nous appellerons les "principes fondamentaux":

Principe du balayage: Quel que soit l'ouvert ω de X et la mesure $\mu \geq 0$ de potentiel $\neq \infty$, il existe une mesure $\mu' \geq 0$ portée par l'adhérence de ω et telle que:

$$G\mu'(x) = G\mu(x) \text{ pour tout } x \in \omega,$$

$$G\mu'(x) \leq G\mu(x) \text{ pour tout } x \in X.$$

Where is the Laplacian in the discrete setting?

The linear operator $\mathcal{L}: \mathcal{C}(V) \rightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y))$$

is called **Combinatorial Laplacian** and the symmetric bilinear

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{V \times V} c(x, y) (u(x) - u(y)) (v(x) - v(y)) dx dy$$

is the **Energy**. Therefore,

$$\mathcal{E}(u, u) = \frac{1}{2} \int_{V \times V} c(x, y) (u(x) - u(y))^2 dx dy$$

The main operators on a network: Schrödinger

If $q \in \mathcal{C}(V)$, the linear operator $\mathcal{L}_q: \mathcal{C}(V) \rightarrow \mathcal{C}(V)$

$$\mathcal{L}_q(u)(x) = \sum_{y \in V} c(x, y)(u(x) - u(y)) + q(x)u(x)$$

is called **Schrödinger operator with potential q** and the symmetric bilinear

$$\mathcal{E}_q(u, v) = \frac{1}{2} \int_{V \times V} c(x, y)(u(x) - u(y))(v(x) - v(y)) dx dy + \int_V q u v dx$$

is the **Energy**. Therefore,

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is the **Energy**. Therefore,

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► For which potentials $q \in \mathcal{C}(V)$ is \mathcal{E}_q positive (semi)-definite?

Schrödinger operators

If $q \in \mathcal{C}(V)$, the linear operator

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is the **Energy**

▶ When $\mathcal{E}_q(u, u) \geq 0$? When $\mathcal{E}_q(u, u) = 0$?

Very easy **sufficient** condition: $q \geq 0$

▶
$$\mathcal{E}_q(u) = \frac{1}{2} \int_{V \times V} c(x, y)(u(x) - u(y))^2 dx dy + \int_V q u^2 dx \geq 0$$

The Laplacian matrix

Let $V = \{x_1, \dots, x_n\}$ and $\Gamma = (V, E, c)$ a connected network where $c(x_i, x_j) = c_{ij}$; i.e., $\{x_i, x_j\} \in E \Leftrightarrow c_{ij} > 0$.

$$\kappa_i = \sum_{j=1}^n c(x_i, x_j) \text{ is the degree of } x_i$$

The irreducible, singular and symmetric **M**-matrix

$$L = \begin{bmatrix} \kappa_1 & -c_{12} & \cdots & -c_{1n-1} & -c_{1n} \\ -c_{12} & \kappa_2 & \cdots & -c_{2n-1} & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{1n-1} & -c_{2n-1} & \cdots & \kappa_{n-1} & -c_{n-1n} \\ -c_{1n} & -c_{2n} & \cdots & -c_{n-1n} & \kappa_n \end{bmatrix}$$

is called the the (combinatorial) **Laplacian** matrix of Γ

Are there M -matrices around here?

Let $V = \{x_1, \dots, x_n\}$ and $\Gamma = (V, E, c)$ a connected network where $c(x_i, x_j) = c_{ij}$; i.e., $\{x_i, x_j\} \in E \Leftrightarrow c_{ij} > 0$.

$$\kappa_i = \sum_{j=1}^n c(x_i, x_j) \text{ is the degree of } x_i, \quad q_i = q(x_i)$$

The irreducible, singular and symmetric \mathbf{Z} -matrix

$$\mathbf{L}_q = \begin{bmatrix} \kappa_1 + q_1 & -c_{12} & \cdots & -c_{1n-1} & -c_{1n} \\ -c_{12} & \kappa_2 + q_2 & \cdots & -c_{2n-1} & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{1n-1} & -c_{2n-1} & \cdots & \kappa_{n-1} & -c_{n-1n} \\ -c_{1n} & -c_{2n} & \cdots & -c_{n-1n} & \kappa_n + q_n \end{bmatrix}$$

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► \mathcal{L}_q is positive semidefinite iff \mathbf{L}_q is an M -matrix

Are there M -matrices around here?

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The irreducible, singular and symmetric \mathbf{Z} -matrix

$$L_q = \begin{bmatrix} \kappa_1 + q_1 & -c_{12} & \cdots & -c_{1n-1} & -c_{1n} \\ -c_{12} & \kappa_2 + q_2 & \cdots & -c_{2n-1} & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{1n-1} & -c_{2n-1} & \cdots & \kappa_{n-1} & -c_{n-1n} \\ -c_{1n} & -c_{2n} & \cdots & -c_{n-1n} & \kappa_n + q_n \end{bmatrix}$$

► Easy sufficient condition: When $q \geq 0 \implies L_q$ is an d.d. M -matrix

A bit of Potential Theory: The Energy Principle

- ▶ For which potentials $q \in \mathcal{C}(V)$ is the energy \mathcal{E}_q positive semidefinite? And positive definite?
- ▶ The Doob potential: $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$, where $\omega \in \Omega(V)$
- ▶
$$q_\omega(x) = -\kappa(x) + \frac{1}{\omega(x)} \sum_{y \in V} c(x, y)\omega(y)$$

A bit of Potential Theory: The Energy Principle

▶ For which potentials $q \in \mathcal{C}(V)$ is the energy \mathcal{E}_q positive semidefinite? And positive definite?

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▶
$$q_\omega(x) = -\kappa(x) + \frac{1}{\omega(x)} \sum_{y \in V} c(x, y)\omega(y)$$

▶ Since $\int_V \omega q_\omega dx = 0$, q_ω takes positive and negative values, except when ω is constant in which case $q_\omega = 0$ and the corresponding Schrödinger operator coincides with the Laplacian.

A bit of Potential Theory: The Energy Principle

▶ For which potentials $q \in \mathcal{C}(V)$ is the energy \mathcal{E}_q positive semidefinite? And positive definite?

▶ The Doob potential: $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$, where $\omega \in \Omega(V)$

$$\text{▶ } q_\omega(x) = -\kappa(x) + \frac{1}{\omega(x)} \sum_{y \in V} c(x, y)\omega(y)$$

▶ Since $\int_V \omega q_\omega dx = 0$, q_ω takes positive and negative values, except when ω is constant in which case $q_\omega = 0$ and the corresponding Schrödinger operator coincides with the Laplacian.

↪ For any proper subset $F \subset V$ there exists $\sigma \in \Omega(V)$ with $\sigma|_F < 0$

The Energy Principle

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$$\mathcal{E}_q(u, u) = \frac{1}{2} \int_{V \times V} c(x, y) \sigma(x) \sigma(y) \left(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)} \right)^2 dx dy + \int_V (q - q_\sigma) u^2 dx$$

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- ▶ \mathcal{E}_q is positive semidefinite and definite iff $q \neq q_\sigma$

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► $\mathcal{C}(V) = \{q_\omega + \lambda : \omega \in \Omega(V), \lambda \in \mathbb{R}\}$: Given $q \in \mathcal{C}(V)$, there exist unique $\omega \in \Omega(V)$ and $\lambda \in \mathbb{R}$ such that $q = q_\omega + \lambda$.

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- ▶ $\min_{\langle u, u \rangle = 1} \{\mathcal{E}_q(u, u)\} = \lambda$ and $\mathcal{E}_q(u, u) = \lambda \Leftrightarrow u = \pm \omega$

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- ▶ \mathcal{E}_q is positive semidefinite iff $\lambda \geq 0$ and definite iff $\lambda > 0$

Application: Characterization of M -matrices

Let $c_{ij} \geq 0$, $1 \leq i < j \leq n$, and the **irreducible** and **symmetric** matrix

$$M = \begin{bmatrix} d_1 & -c_{12} & \cdots & -c_{1n-1} & -c_{1n} \\ -c_{12} & d_2 & \cdots & -c_{2n-1} & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{1n-1} & 0 & \cdots & d_{n-1} & -c_{n-1n} \\ -c_{1n} & 0 & \cdots & -c_{n-1n} & d_n \end{bmatrix}.$$

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M is an M -matrix iff there exists $\omega \in \Omega(V)$ and $\lambda \geq 0$ such that

$$d_i = \lambda + \frac{1}{\omega_i} \sum_{j=1}^n c_{ij} \omega_j.$$

Moreover, M is invertible iff $\lambda > 0$.

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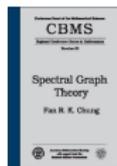
► Moreover, M is invertible iff $\lambda > 0$. Equivalently, iff there exists

$$\sigma \in \Omega(V) \text{ such that } d_i \geq \frac{1}{\sigma_i} \sum_{j=1}^n c_{ij} \sigma_j$$

Application: Normalized Laplacian

$$\Delta(u)(x) = \frac{1}{\sqrt{\kappa(x)}} \sum_{y \in V} c(x, y) \left(\frac{u(x)}{\sqrt{\kappa(x)}} - \frac{u(y)}{\sqrt{\kappa(y)}} \right)$$

↪ Introduced in



Popularized in

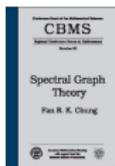


F.R.K. Chung & S.T. Yau
Discrete Green's functions,
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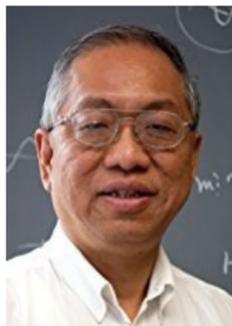


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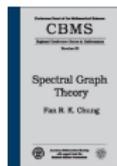


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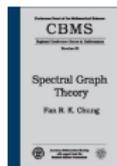
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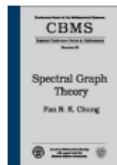
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▶ Consider the network $\hat{\Gamma} = (V, E, \hat{c})$ where $\hat{c} = \frac{c(x, y)}{\sqrt{\kappa(x)}\sqrt{\kappa(y)}}$

▶ $\Delta = \hat{\mathcal{L}}_{q\omega}$ where $\omega = \frac{1}{\text{vol}(\Gamma)} \sqrt{\kappa}$ and $\text{vol}(\Gamma) = \sum_{x \in V} \kappa(x)$

The main operator on a network: The Laplacian

The linear operator $\mathcal{L}: \mathcal{C}(V) \rightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y))$$

is called **Combinatorial Laplacian** and the symmetric bilinear

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{V \times V} c(x, y) (u(x) - u(y)) (v(x) - v(y)) dx dy$$

is the **Energy**. Therefore,

$$\mathcal{E}(u, u) = \frac{1}{2} \int_{V \times V} c(x, y) (u(x) - u(y))^2 dx dy$$

The matrix L is called the the **(combinatorial) Laplacian matrix** of Γ

↪ **Why** \mathcal{L} and L are called **Laplacian**?

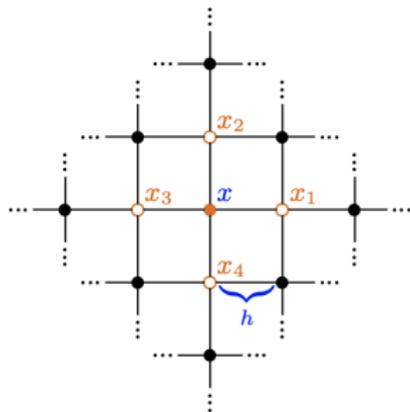
Difference Schemes

- Differential operator:

$$\mathcal{L}(u) = -(k_1 u_{xx} + k_2 u_{yy}), \quad k_1, k_2 > 0$$

- Stencil:

$$S_h(\mathbf{x}) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$$



- Five Points Formula:

$$L^h(u)(\mathbf{x}) = \frac{1}{h^2} \left[k_1 (2u(\mathbf{x}) - u(\mathbf{x}_1) - u(\mathbf{x}_3)) + k_2 (2u(\mathbf{x}) - u(\mathbf{x}_2) - u(\mathbf{x}_4)) \right]$$

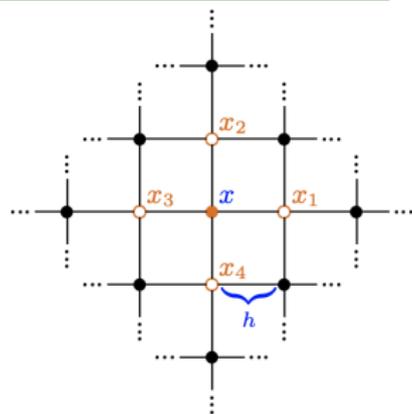
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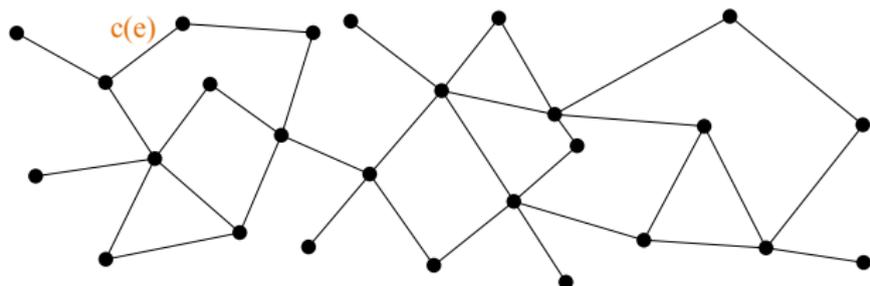
$$\begin{aligned} L^h(u)(x) &= \frac{k_1}{h^2} (u(x) - u(x_1)) + \frac{k_2}{h^2} (u(x) - u(x_2)) \\ &\quad + \frac{k_2}{h^2} (u(x) - u(x_3)) + \frac{k_1}{h^2} (u(x) - u(x_4)) \end{aligned}$$

The Laplacian matrix

Let the **irreducible** and **symmetric** matrix

$$L = \begin{bmatrix} \kappa_1 & -c_{12} & \cdots & -c_{1n-1} & -c_{1n} \\ -c_{12} & \kappa_2 & \cdots & -c_{2n-1} & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{1n-1} & 0 & \cdots & \kappa_{n-1} & -c_{n-1n} \\ -c_{1n} & 0 & \cdots & -c_{n-1n} & \kappa_n \end{bmatrix} \in \mathcal{M}_{|V| \times |V|}(\mathbb{R})$$

► Consider $C = (c(e)) \in \mathcal{M}_{|E| \times |E|}(\mathbb{R})$ the **conductance matrix**

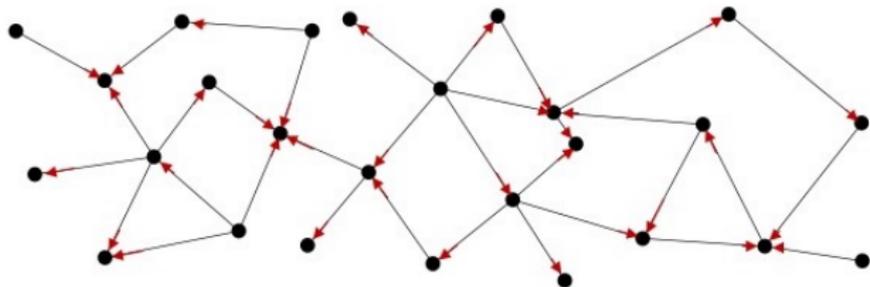


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Incidence Matrix

$$e \rightarrow \begin{bmatrix} x & y & z & w \\ 1 & 0 & \cdots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{e} \rightarrow [-1 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = K \in \mathcal{M}_{|E| \times |V|}(\mathbb{R})$$

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- ▶ Consider $\mathbf{C} = (c(e)) \in \mathcal{M}_{|E| \times |E|}(\mathbb{R})$ the **conductance** matrix
- ▶ Consider an **arbitrary** orientation on Γ and $\mathbf{K} \in \mathcal{M}_{|E| \times |V|}(\mathbb{R})$ its **incidence** matrix $\implies \mathbf{L} = \mathbf{K}^\top \mathbf{C} \mathbf{K}$

$$\begin{bmatrix} x & y & z & w \\ \mathbf{1} & 0 & \cdots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{1} & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u(x) \\ \vdots \\ u(w) \end{bmatrix} = \begin{bmatrix} u(x) - u(z) \\ \vdots \\ u(w) - u(x) \end{bmatrix} \begin{matrix} \leftarrow e \\ \\ \leftarrow \hat{e} \end{matrix}$$

The Laplacian matrix

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- ▶ Consider an **arbitrary orientation** on Γ and $K \in \mathcal{M}_{|E| \times |V|}(\mathbb{R})$
- ▶ The **Laplacian** satisfies that $L = \underbrace{K^T}_{\text{div}} \overbrace{C \quad K}^{-\nabla} \underbrace{\quad}_{-d}$
- ▶ If $u \in \mathbb{R}^{|V|} \equiv \mathcal{C}(V) \implies Lu = -\text{div}(\nabla(u))$

Back to Mimetic Methods

Mimetic Methods are based on vector calculus on weighted graphs.

↪ Its main ingredients are functions and vector fields, related each other through discrete difference operators.

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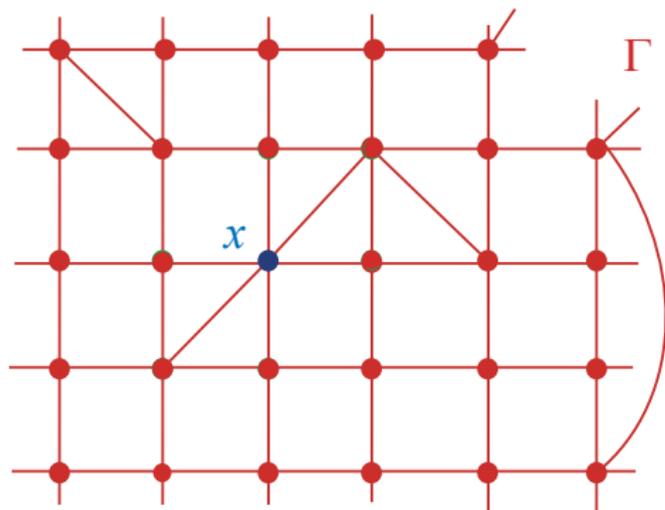
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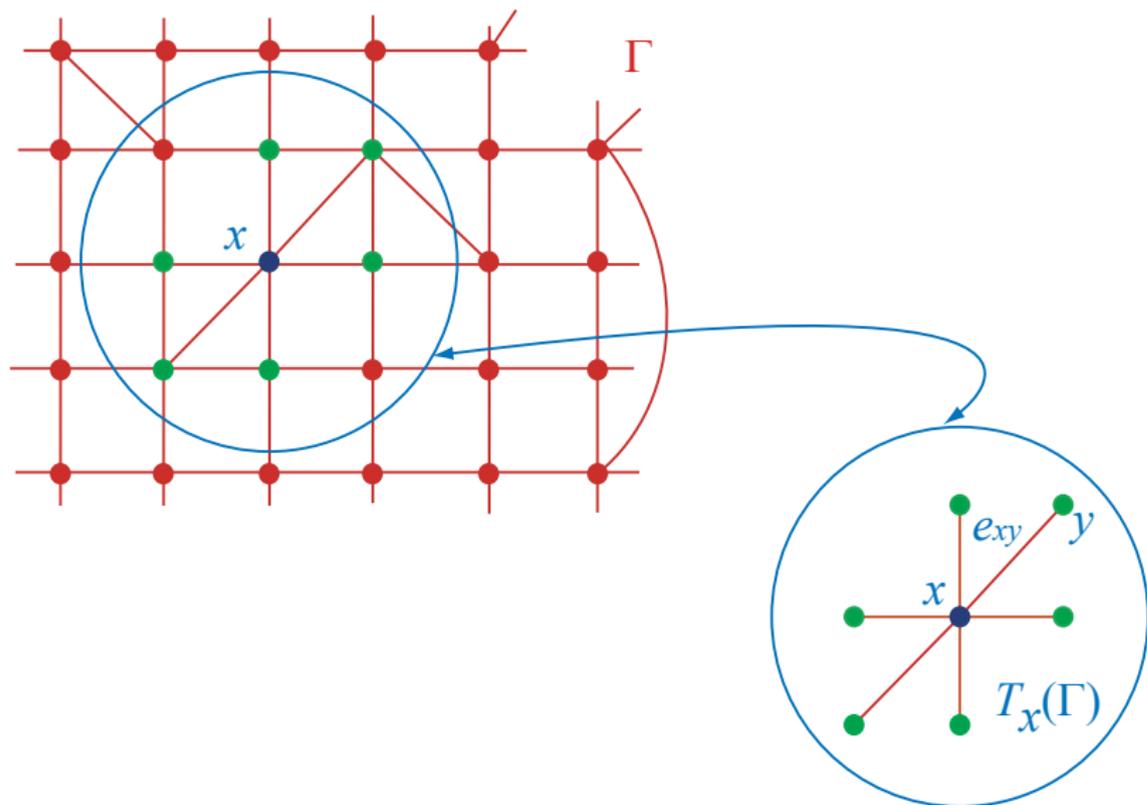
▶ If they exist, do these operators mimetize the differential ones ?

$$\mathcal{K} = -\mathcal{F}^*, \quad \mathcal{H}^* = \mathcal{H}, \quad \mathcal{K} \circ \mathcal{F} = \text{Laplacian}, \quad \mathcal{H} \circ \mathcal{F} = 0, \quad \mathcal{K} \circ \mathcal{H} = 0$$

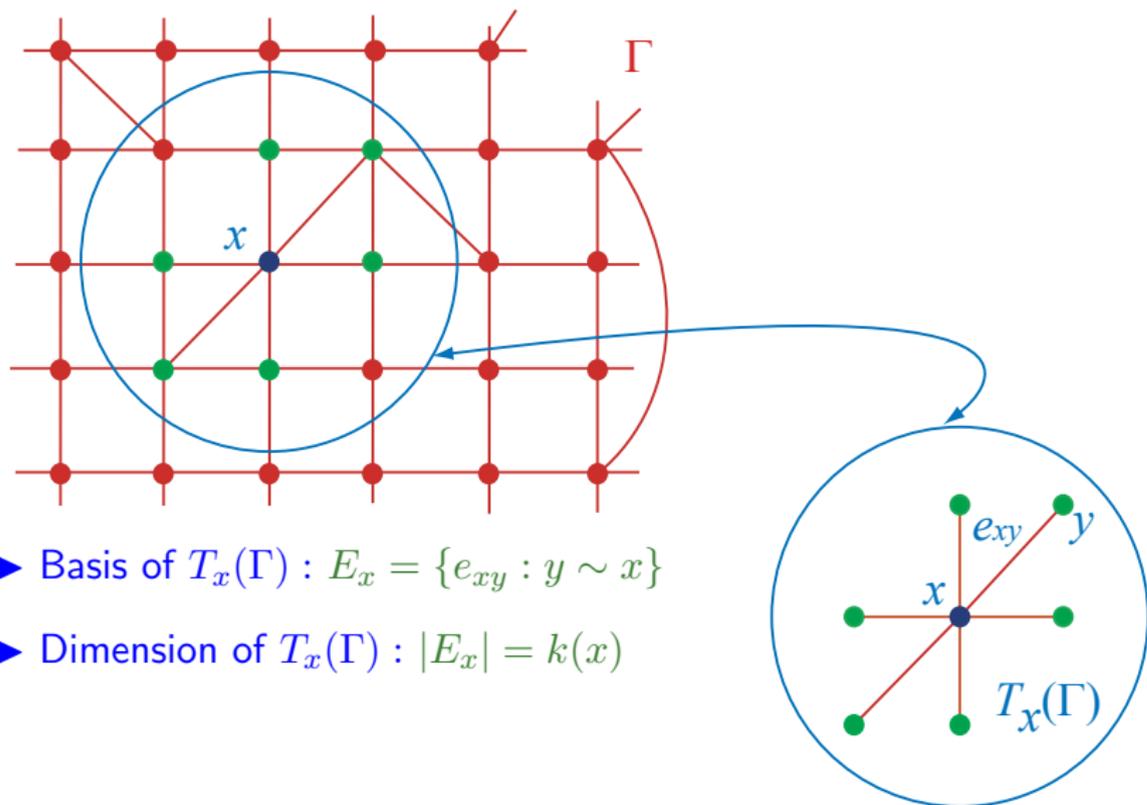
Our proposal: The Tangent space



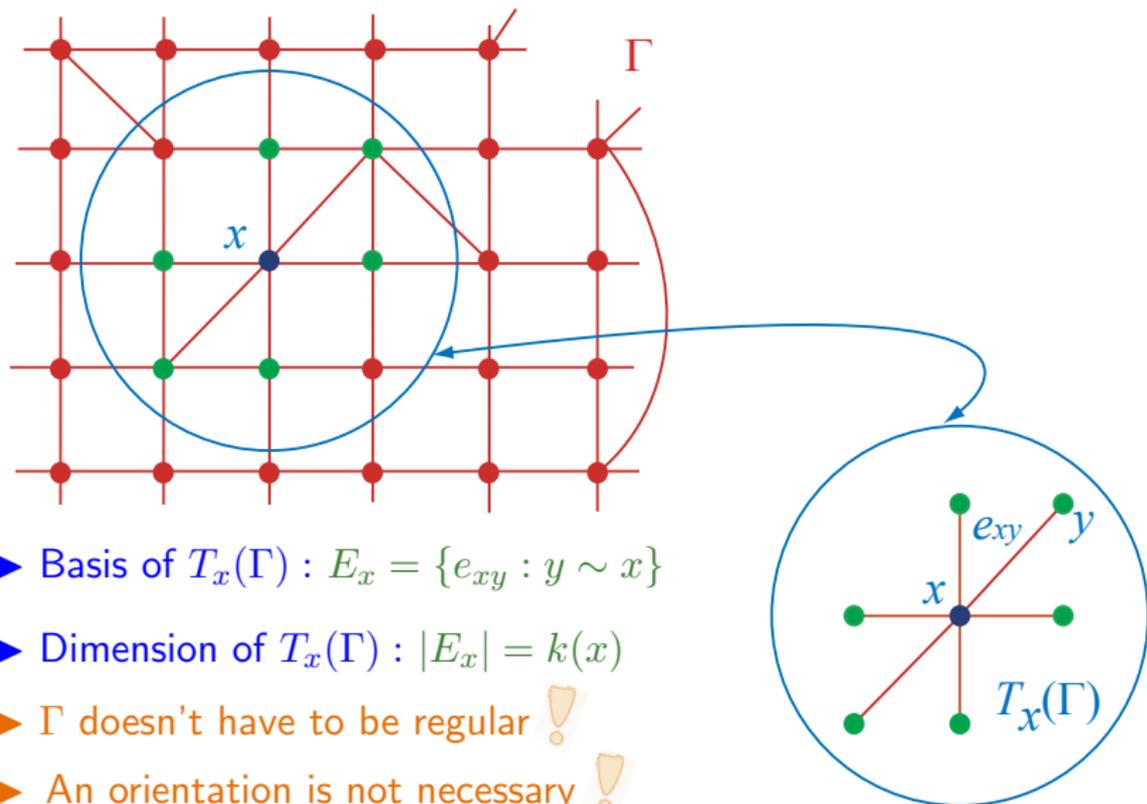
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Vector fields on a finite graph

- ▶ Vector field: $f : V \longrightarrow \bigcup_{x \in V} T_x(\Gamma)$ such that $f(x) \in T_x(\Gamma)$
- ▶ Space of vector fields: $\mathcal{X}(\Gamma)$
- ▶ Component function: $f(x) = \sum_{y \sim x} f(x, y)e_{xy}$, $f \in \mathcal{C}(\Gamma)$

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► f is a symmetric field: $f(x, y) = f(y, x)$, $x, y \in V$

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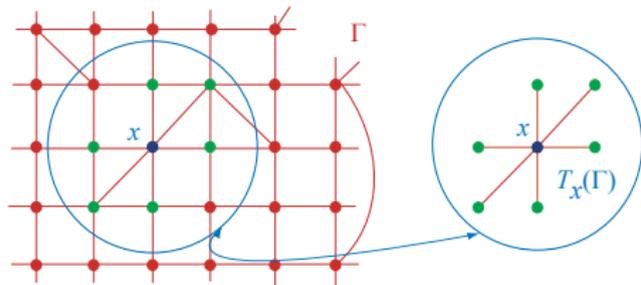
▶ Its component function is

$$g(x, y) = \sum_{z \in V} m(x, y, z) f(x, z)$$

First order operators

► Connected graph $\Gamma = (V, E)$ and weight $\nu \in \Omega(V)$

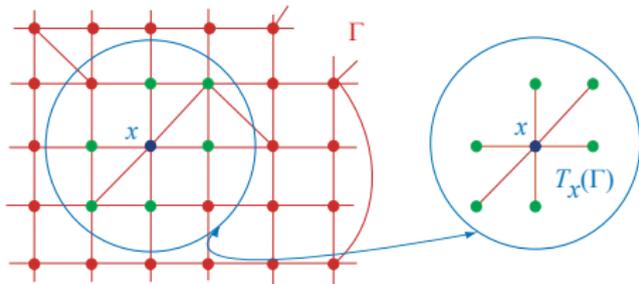
► Inner products on $\mathcal{C}(V)$ and $\mathcal{X}(\Gamma)$: $\int_V uv d\nu$ and $\frac{1}{2} \int_V (f, g) dx$



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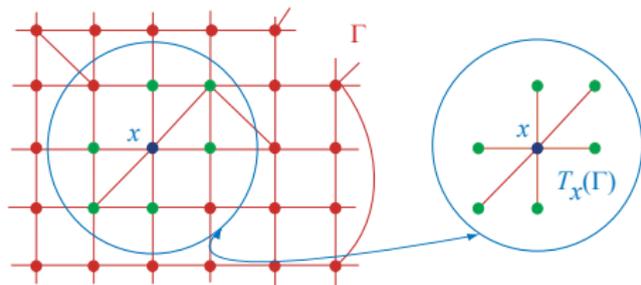


★ Derivative $d: \mathcal{C}(V) \longrightarrow \mathcal{X}(\Gamma) \implies du(x) = \sum_{y \sim x} (u(y) - u(x)) e_{xy}$

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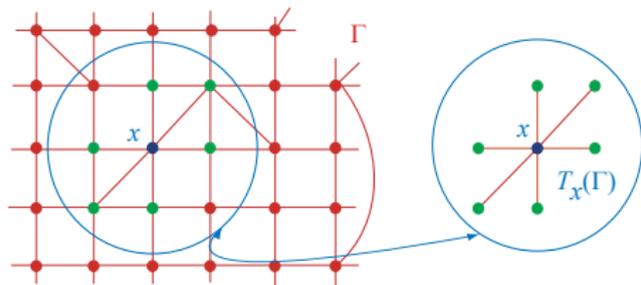
★ Divergence $\text{div}: \mathcal{X}(\Gamma) \longrightarrow \mathcal{C}(V) \implies \text{div} = -d^*$

$$\int_V u \text{div}(f) d\nu = -\frac{1}{2} \int_V (du, f) dx, \quad u \in \mathcal{C}(V)$$

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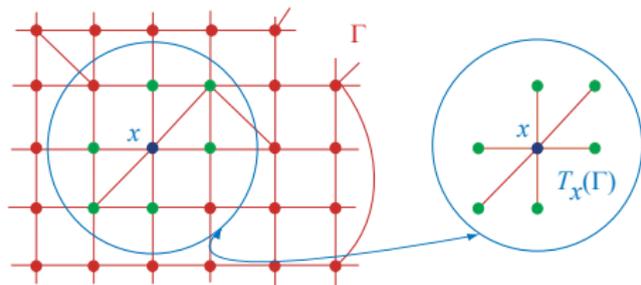
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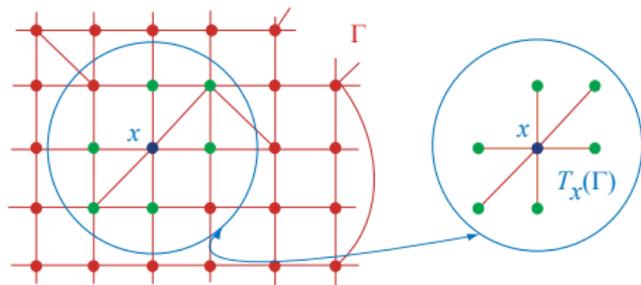
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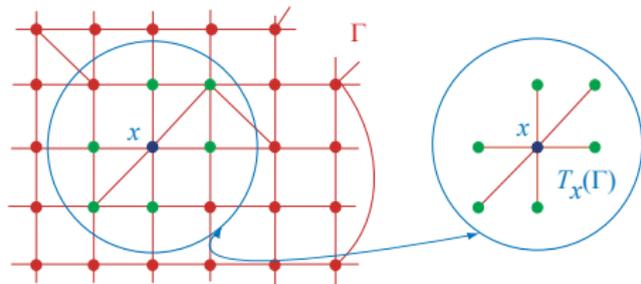
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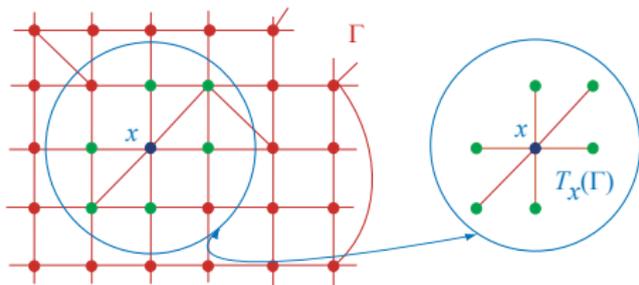
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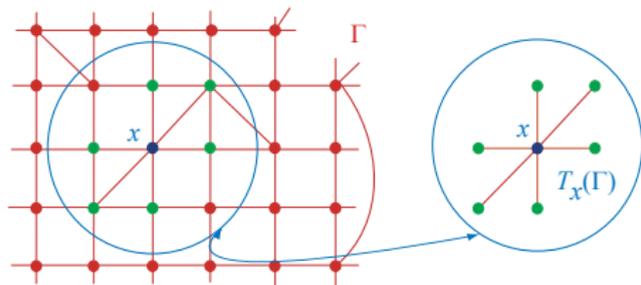
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\rightsquigarrow When M is symmetric and positive definite, d_M is the gradient
for the metric $G = M^{-1} \implies \nabla_G$

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↪ $\text{curl}: \mathcal{X}(\Gamma) \rightarrow \mathcal{X}(\Gamma)$, $\text{curl}(f) = (Gf)^s$

First order operators

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- ▶ Metric on Γ : G
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- ▶ Derivative of $u \in \mathcal{C}(V)$: $d(u) \in \mathcal{X}^a(\Gamma)$
- ▶ Gradient of $u \in \mathcal{C}(V)$, respect to G : $\nabla_G(u) = G^{-1}d(u)$
- ▶ Divergence of $f \in \mathcal{X}(\Gamma)$: $\operatorname{div}(f)(x) = \nu(x)^{-1} \sum_{y \in V} f^a(x, y)$
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★ Mimetic properties:

$$\operatorname{div} = -\nabla_G^*, \quad \operatorname{div} \circ \operatorname{curl} = 0, \quad \operatorname{curl} \circ \nabla_G = 0, \quad \operatorname{curl}^* = \operatorname{curl}$$

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★ Mimetic properties: De Rham's complex:

$$0 \xrightarrow{0} \mathcal{C}(V) \xrightarrow{\nabla_G} \mathcal{X}(\Gamma) \xrightarrow{\operatorname{curl}} \mathcal{X}(\Gamma) \xrightarrow{\operatorname{div}} \mathcal{C}(V) \xrightarrow{0} 0$$

Second order operators

- ▶ Connected graph $\Gamma = (V, E)$ and weight $\nu \in \Omega(V)$
- ▶ Metric on Γ : G and $m \in \mathcal{C}(\Gamma \times \Gamma)$ the component of G^{-1}
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- ★ Laplace-Beltrami: $\Delta: \mathcal{C}(V) \rightarrow \mathcal{C}(V)$, $\Delta = -\operatorname{div} \circ \nabla$
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or more generally $\mathcal{L}(u) = -\operatorname{div}(Md(u))$ for a symmetric matrix field M

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \sum_{y \in V} \mathbf{c}(x, y) (u(x) - u(y)), \quad x \in V$$

where $c(x, x) = 0$, $x \in V$ and for $x, y \in V$ with $x \neq y$,

$$\mathbf{c}(x, y) = \frac{1}{2} \sum_{z \in V} [m(x, y, z) + m(y, x, z) - m(z, x, y)]$$

Elliptic second order operators

► Connected graph $\Gamma = (V, E)$ and weight $\nu \in \Omega(V)$

★ $\mathcal{L}(u) = -\text{div}(\text{Md}(u))$ for a symmetric matrix field M

► $\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \sum_{y \in V} c(x, y)(u(x) - u(y)), x \in V$

► $c(x, y) = \frac{1}{2} \sum_{z \in V} [m(x, y, z) + m(y, x, z) - m(z, x, y)]$

① c is symmetric: $c(x, y) = c(y, x)$

② $c(x, y) = 0$ if $d(x, y) > 2$

③ $c(x, y) = -\frac{1}{2} \sum_{z \in V} \mu(z)m(z, x, y)$ if $d(x, y) = 2$

④ $c(x, y) = \frac{1}{2} \sum_{z \in V} [m(x, y, z) + m(y, x, z)]$ if

$d(x, y) = 1$ and x, y don't belong to any triangle

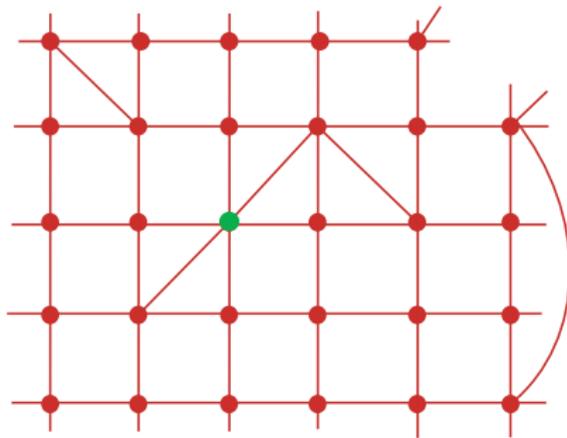
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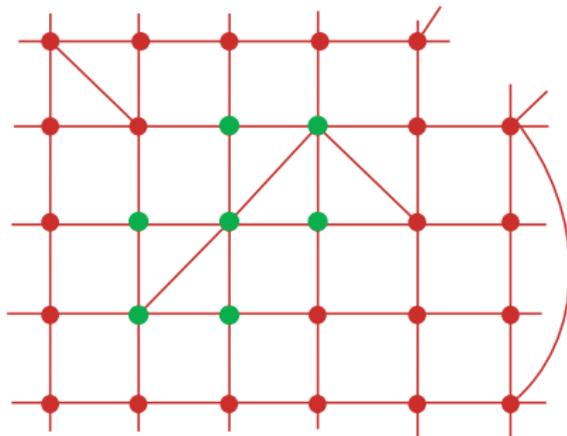
Elliptic second order operators

▶ Connected graph $\Gamma = (V, E)$ and weight $\nu \in \Omega(V)$

★ $\mathcal{L}(u) = -\text{div}(\text{Md}(u))$ for a symmetric matrix field M

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$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \sum_{y \in V} c(x, y)(u(x) - u(y)), \quad x \in V$$

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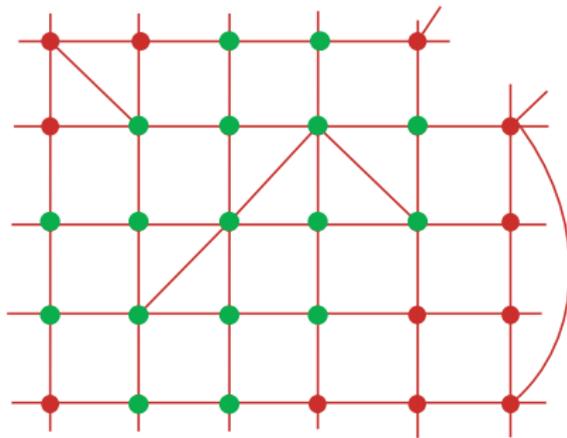
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▶ Given $u, v \in \mathcal{C}(V)$,

$$\int_V u \mathcal{L}(v) d\nu = \int_{V \times V} c(x, y)(u(x) - u(y))(v(x) - v(y)) dx dy$$

$$\rightsquigarrow \mathcal{L} \text{ is self-adjoint: } \int_V u \mathcal{L}(v) d\nu = \int_V v \mathcal{L}(u) d\nu$$

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↪ Semi-Ellipticity: When $\int_V u \mathcal{L}(u) d\nu \geq 0$?

↪ Ellipticity: When $\int_V u \mathcal{L}(u) d\nu = 0$ implies u constant ?

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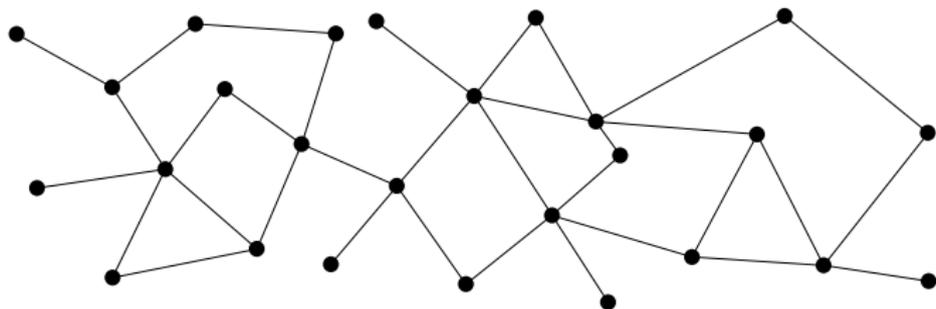
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▶ Semi-ellipticity when $M(x)$ is a d.d. M -matrix for any $x \in V$

Boundary of a set

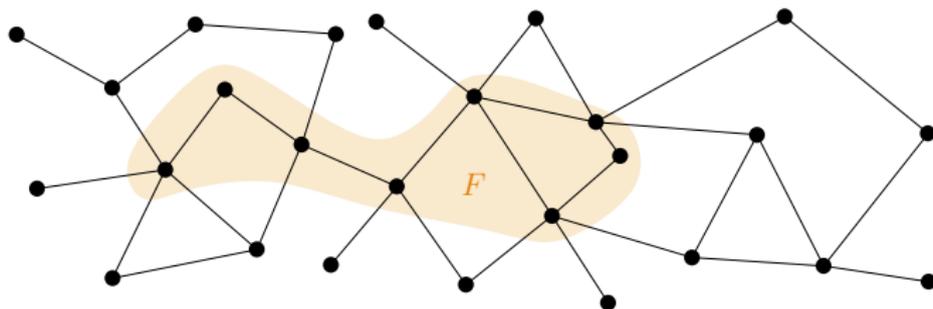
Consider a Graph $\Gamma = (V, E)$: V vertices, E edges



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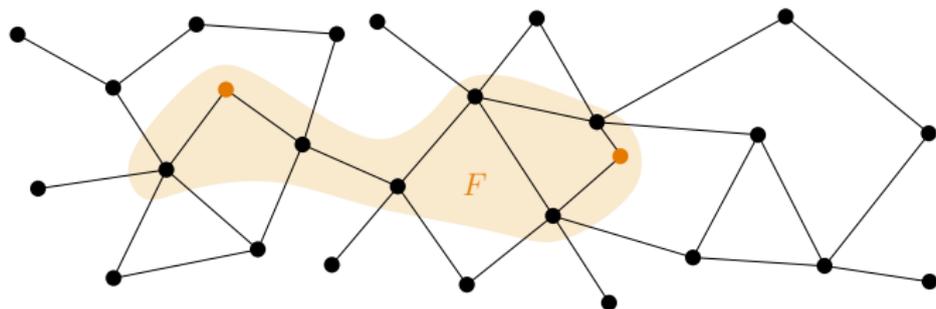
Consider $F \subset V$ a non-empty set



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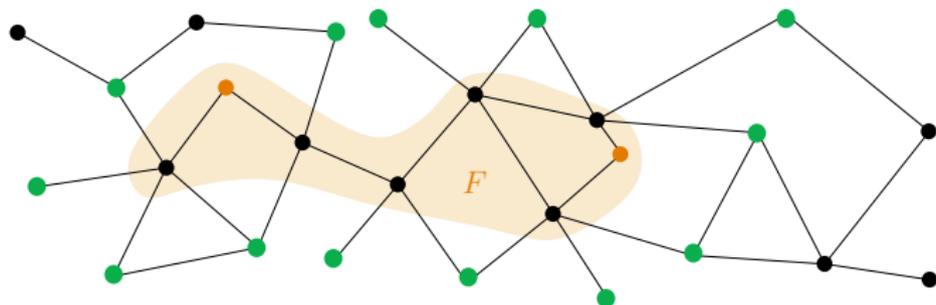
$$\overset{\circ}{F} = \{x \in F : V(x) \subset F\}$$

interior of F

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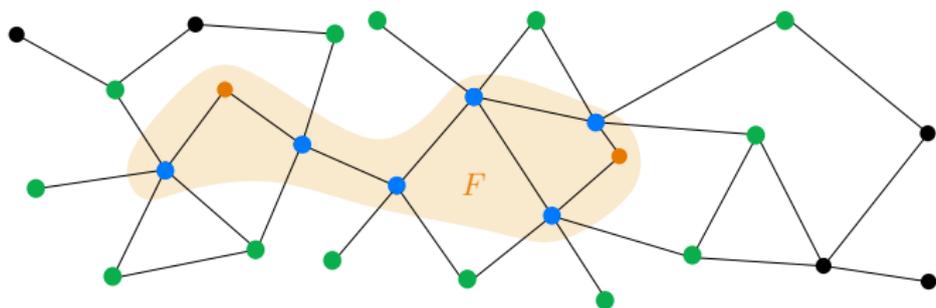
$$\overset{\circ}{F} = \{x \in F : V(x) \subset F\} \quad \text{interior of } F$$

$$\delta(F) = \{x \in F^c : F \cap V(x) \neq \emptyset\} \quad \text{boundary of } F$$

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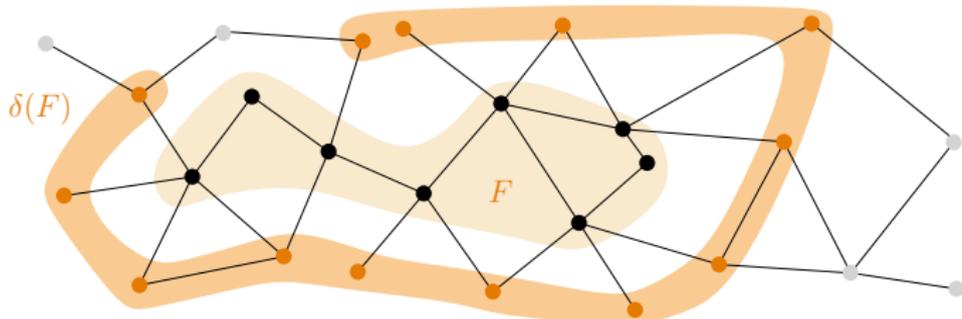
$$\delta(F) = \{x \in F^c : F \cap V(x) \neq \emptyset\} \quad \text{boundary of } F$$

$$\delta(F^c) = F \setminus \overset{\circ}{F} \quad \text{interior boundary of } F$$

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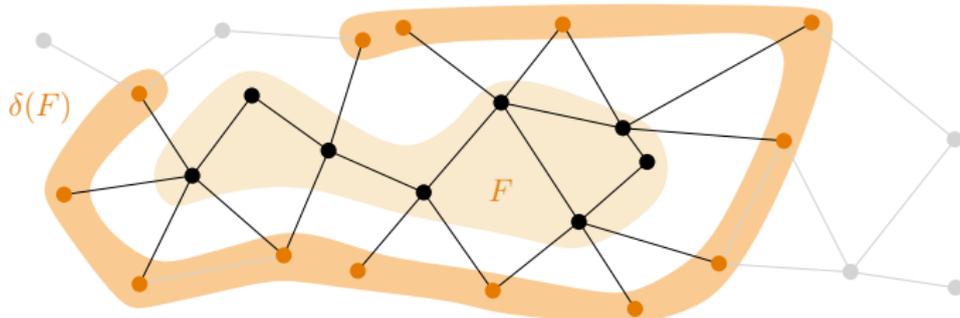
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$$\bar{F} = F \cup \delta(F) \quad \text{closure of } F$$

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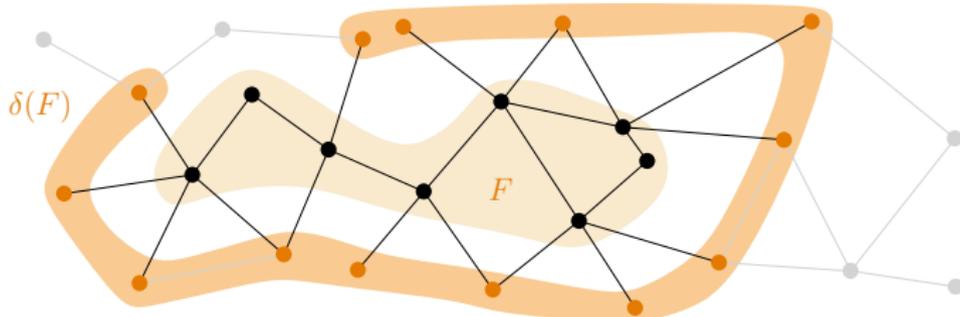
$$\bar{F} = F \cup \delta(F) \quad \text{closure of } F$$

- Delete the vertices not in \bar{F} and the edges whose both extremes are out of F .

Boundary of a set

Consider a Network $\Gamma = (V, E, c)$: V vertices, E edges

Consider $F \subset V$ a non-empty set



$$\overset{\circ}{F} = \{x \in F : V(x) \subset F\} \quad \text{interior of } F$$

$$\delta(F) = \{x \in F^c : F \cap V(x) \neq \emptyset\} \quad \text{boundary of } F$$

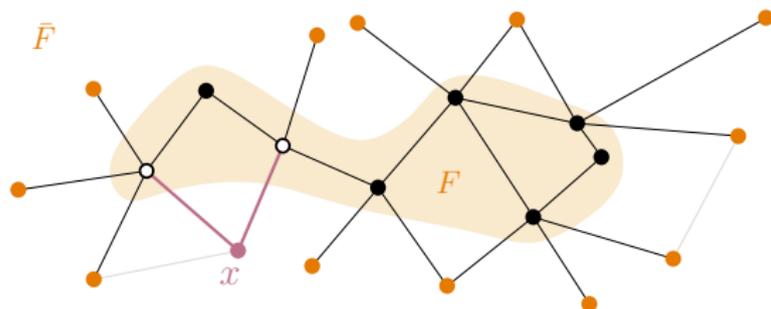
$$\bar{F} = F \cup \delta(F) \quad \text{closure of } F$$

- Delete the vertices not in \bar{F} and the edges whose both extremes are out of F . In the sequel $c \equiv c \cdot \chi_{(\bar{F} \times \bar{F}) \setminus (\delta(F) \times \delta(F))}$ when F is fixed.

Normal vector field

Consider a Graph $\Gamma = (V, E)$: V vertices, E edges

Consider $F \subset V$ a non-empty set and χ_F its **characteristic function**

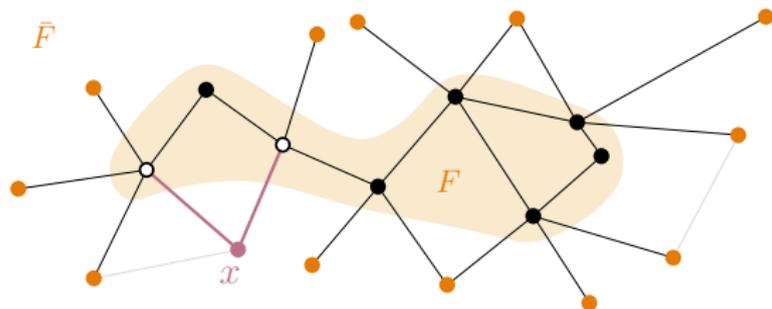


Normal vector field to F : $n_F = -d\chi_F$

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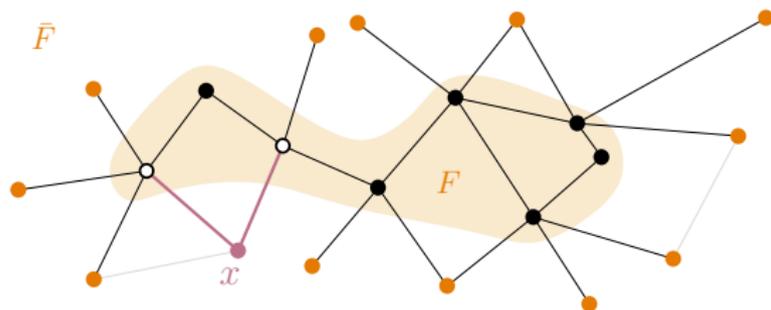
and hence $d\chi_F(x) = \sum_{y \sim x} (\chi_F(y) - \chi_F(x))e_{xy}$

$$\rightsquigarrow n_F(x, y) = \begin{cases} -1, & \text{when } x \sim y \text{ and } (x, y) \in \delta(F) \times \delta(F^c) \\ 1, & \text{when } x \sim y \text{ and } (x, y) \in \delta(F^c) \times \delta(F) \\ 0, & \text{otherwise} \end{cases}$$

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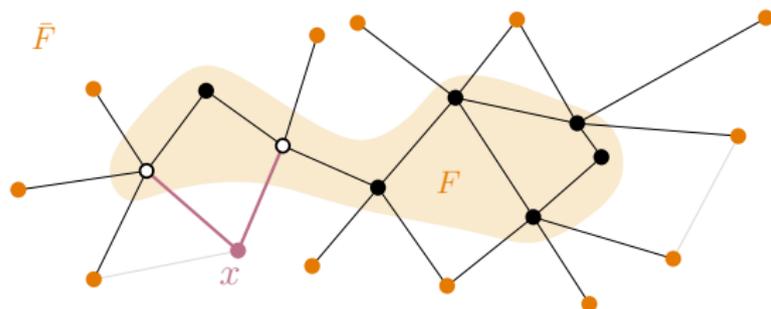
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► n_F is a flow, $n_{F^c} = -n_F$ and $\text{supp}(n_F) = \delta(F) \cup \delta(F^c)$

Schrödinger operators and normal derivatives

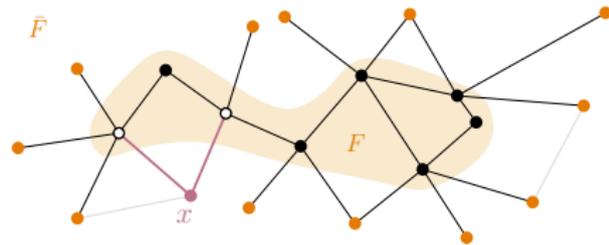
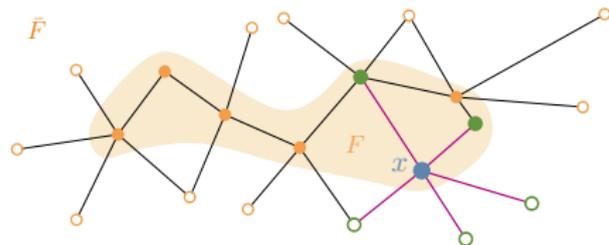
Let $q \in \mathcal{C}(\bar{F})$, then

$$\mathcal{L}_q(u)(x) = \sum_{y \in \bar{F}} c(x, y)(u(x) - u(y)) + q(x)u(x)$$

is called **Schrödinger operator** with **potencial** q , whereas

$$\frac{\partial u}{\partial \mathbf{n}_F}(x) = (\mathbf{n}_F, \nabla u) = \sum_{y \in F} c(x, y)(u(x) - u(y)), \quad x \in \delta(F)$$

is called **normal derivative**.



Green Identities

Let $q \in \mathcal{C}(\bar{F})$ and $u, v \in \mathcal{C}(\bar{F})$

First Green Identity

$$\int_F v \mathcal{L}_q(u) dx = \frac{1}{2} \int_{\bar{F} \times \bar{F}} c(x, y) (u(x) - u(y)) (v(x) - v(y)) dx dy \\ + \int_F q u v dx - \int_{\delta(F)} v \frac{\partial u}{\partial \mathbf{n}_F} dx$$

Second Green Identity

$$\int_F \left[v \mathcal{L}_q(u) - u \mathcal{L}_q(v) \right] dx = \int_{\delta(F)} \left[u \frac{\partial v}{\partial \mathbf{n}_F} - v \frac{\partial u}{\partial \mathbf{n}_F} \right] dx$$

Gauss' Theorem

$$\int_F \mathcal{L}(u) dx = - \int_{\delta(F)} \frac{\partial u}{\partial \mathbf{n}_F} dx$$

Self-Adjoint Boundary Value Problems

Let $\delta(F) = H_1 \cup H_2$ a partition, $q \in \mathcal{C}(F \cup H_1)$, $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$ and $g_2 \in \mathcal{C}(H_2)$. A **boundary value problem on F** consists on finding $u \in \mathcal{C}(\bar{F})$ such that

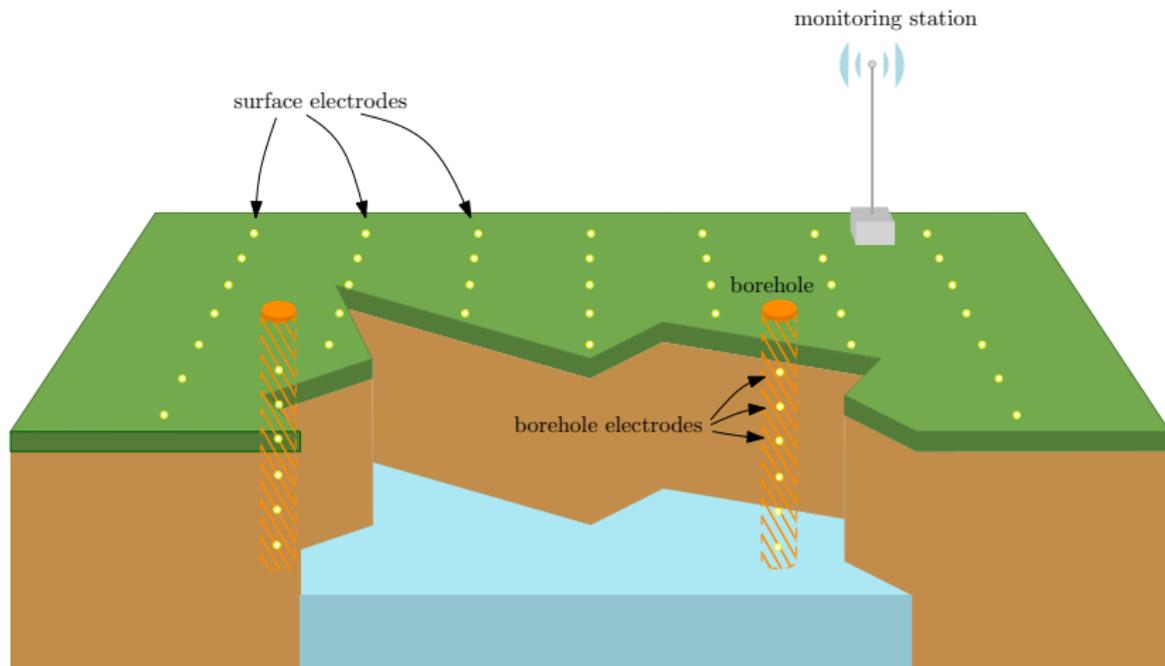
$$\mathcal{L}_q(u) = g \text{ on } F, \quad \frac{\partial u}{\partial \mathbf{n}_F} + qu = g_1 \text{ on } H_1 \text{ and } u = g_2 \text{ on } H_2$$

It is generically called **Dirichlet–Robin Problem**

- **Dirichlet Problem** $\emptyset \neq H_2 = \delta(F)$
- **Robin Problem** $\emptyset \neq H_1 = \delta(F)$ and $q \neq 0$ on H_1
- **Neumann Problem** $\emptyset \neq H_1 = \delta(F)$ and $q = 0$ on H_1
- **Dirichlet–Neumann Problem** $H_1, H_2 \neq \emptyset$ and $q = 0$ on H_1
- **Poisson equation** $H_1, H_2 = \emptyset$

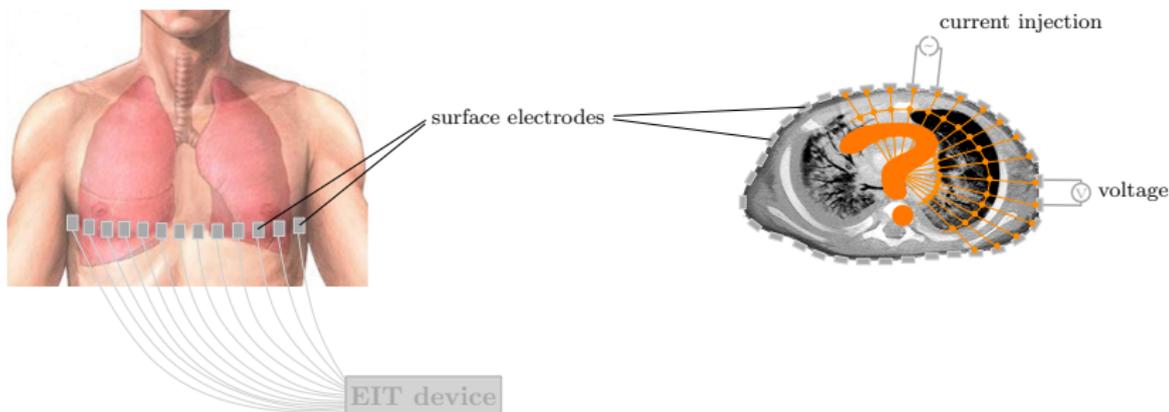
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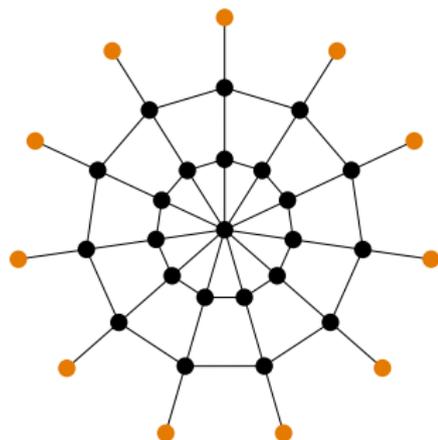
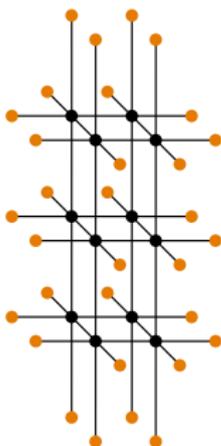
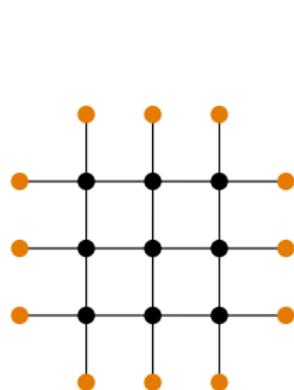
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- ▶ They have several **practical applications** such as **Electrical Impedance Tomography**
- ▶ Have been extensively studied in the **continuous field**; in the **discrete setting** they are more **recent**
- ▶ Can be divided into several parts:
 - **Identifiability: Injectivity** of the inverse map.
 - **Stability: Continuity** of the inverse map.
 - **Characterization: Range** of the inverse map.
 - **Reconstruction: Recovering formula** of the conductance and/or **numerical algorithms** to approximate the conductance.

Inverse Problems

- ▶ **Objective:** To recover the conductivity function (**unknown**) using only **boundary measurements** and global **equilibrium conditions**
- ▶ **Difficulty:** This problem is exponentially **ill-posed**. It is highly sensitive to changes in the boundary data
- ▶ **What we do:** Find situations where the **recovery is feasible**



Dirichlet-to-Neumann Map

Let $F \subset V$ be a proper connected subset. For any $g \in \delta(F)$, consider the **Dirichlet problem**

$$\mathcal{L}(u) = 0 \text{ on } F, \quad u = g \text{ on } \delta(F)$$

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- ▶ There exists a unique solution $u_g \in \mathcal{C}(\bar{F})$

The map $\Lambda: \mathcal{C}(\delta(F)) \rightarrow \mathcal{C}(\delta(F))$, defined as

$$\Lambda(g) = \frac{\partial u_g}{\partial \mathbf{n}_F}$$

▶ is the D-to-N map. Its kernel $N: \delta(F) \times \delta(F) \rightarrow \mathbb{R}$ is the Response kernel (matrix) of the set \bar{F} .

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▶ $\int_{\delta(F)} g \Lambda(g) = \mathcal{E}(u_g, u_g)$

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$$\text{If } M = \begin{bmatrix} D(\delta(F), \delta(F)) & -C(\delta(F), F) \\ -C(F, \delta(F)) & D(F, F) - C(F, F) \end{bmatrix} = \begin{bmatrix} D & -C \\ -C^\top & L_F \end{bmatrix},$$

$$N = M/L_F = D - CL_F^{-1}C^\top \implies N \text{ is a symmetric } M\text{-matrix}$$

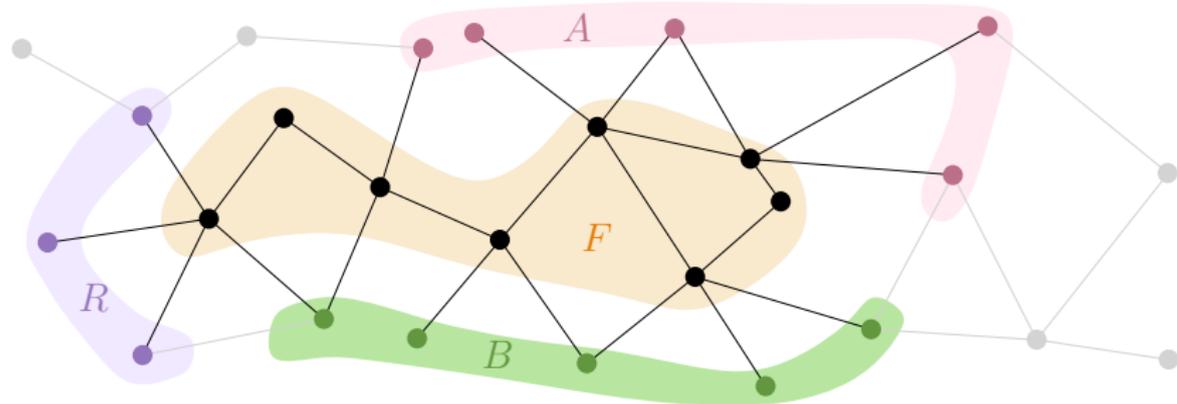
Overdetermined partial BVP

Let $F \subset V$ be a proper connected set and consider $\delta(F) = A \cup B \cup R$ a partition of $\delta(F)$. Unlike standard BVP on $\bar{F} = F \cup \delta(F)$ we assume to have

on R simple information

on A double information

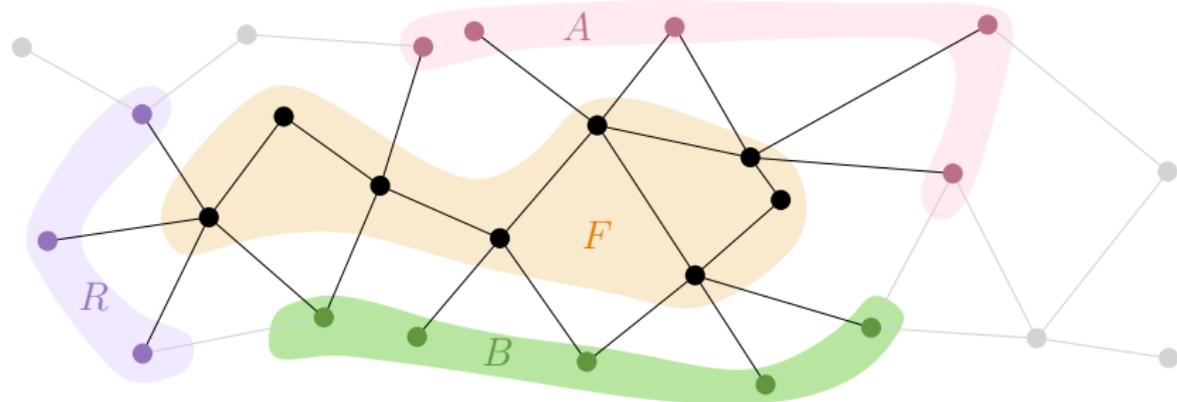
on B no information at all!



Overdetermined partial BVP

Let $F \subset V$ be a proper connected set and consider $\delta(F) = A \cup B \cup R$ a partition of $\delta(F)$. Given $f \in \mathcal{C}(F)$, $h \in \mathcal{C}(A)$ and $g \in \mathcal{C}(A \cup R)$ the **overdetermined partial Dirichlet–Neumann BVP on F** with data f, g and h consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}(u) = f \text{ on } F, \quad \frac{\partial u}{\partial n_F} = h \text{ on } A, \quad u = g \text{ on } A \cup R$$



Overdetermined partial BVP

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When $|A| = |B|$, then for any $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the overdetermined partial Dirichlet–Neumann BVP on F with data $f = 0$, g and h , **has a unique solution iff $N(B; A)$ is invertible**. In this case,

$$u = N(B; A)^{-1}h - N(B; A)^{-1} \cdot N(A \cup R; A)g \text{ on } B$$

**THANK YOU FOR
YOUR ATENTION**

