VI Jornadas ALAMA: Laplacian and M-matrices on Graphs
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Spherical Euclidean matrices and effective resistance in graphs

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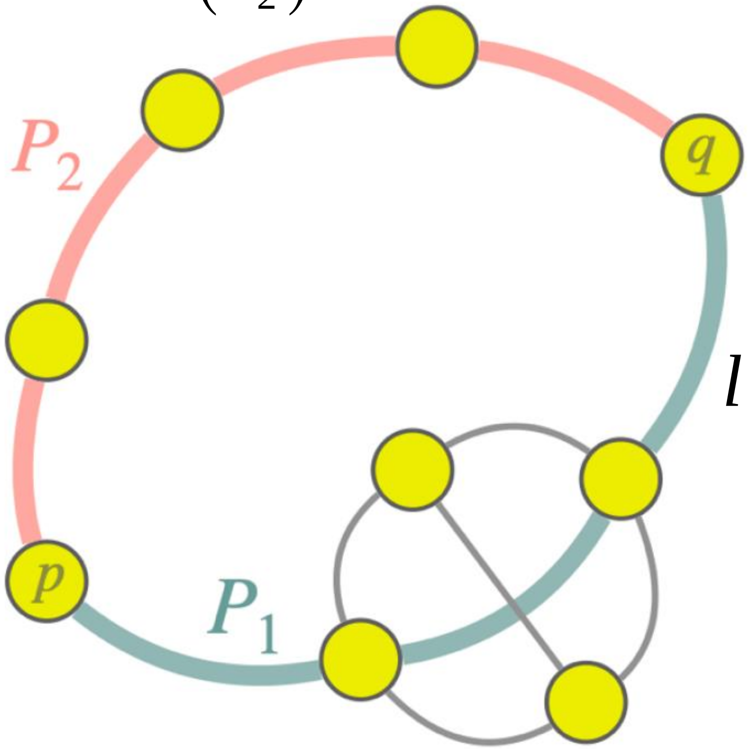
Motivation



Hitting time: the time (number of steps) that a random walker, departing from p, takes to hit by the first time the node q.

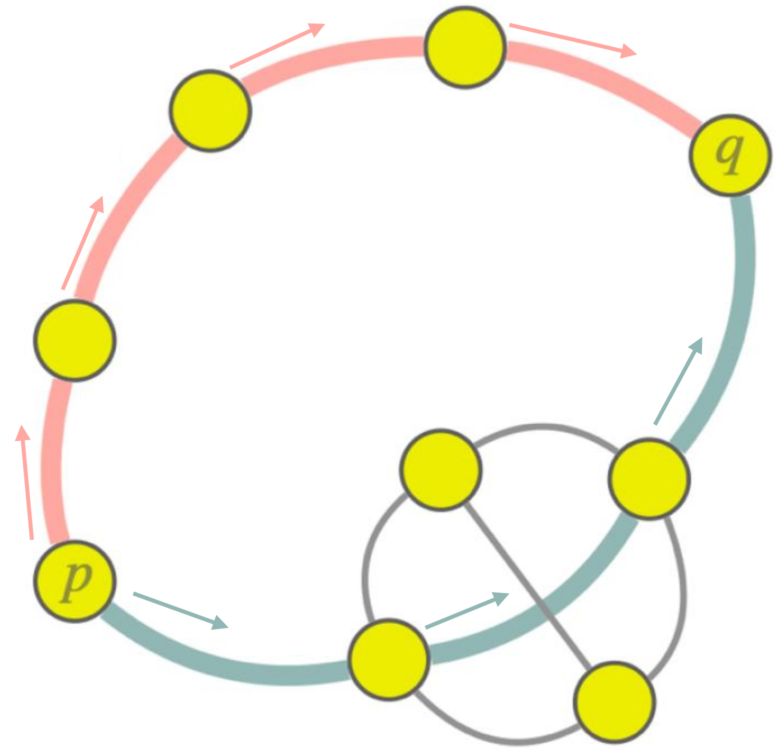
Excess time: the time (number of steps) taken by a random walker, departing from p, to arrive to q in comparison with a ballistic walk.

	P_1	P_2
Hitting time	**10.9**	**6.9**
Excess time	**2.7**	**1.38**

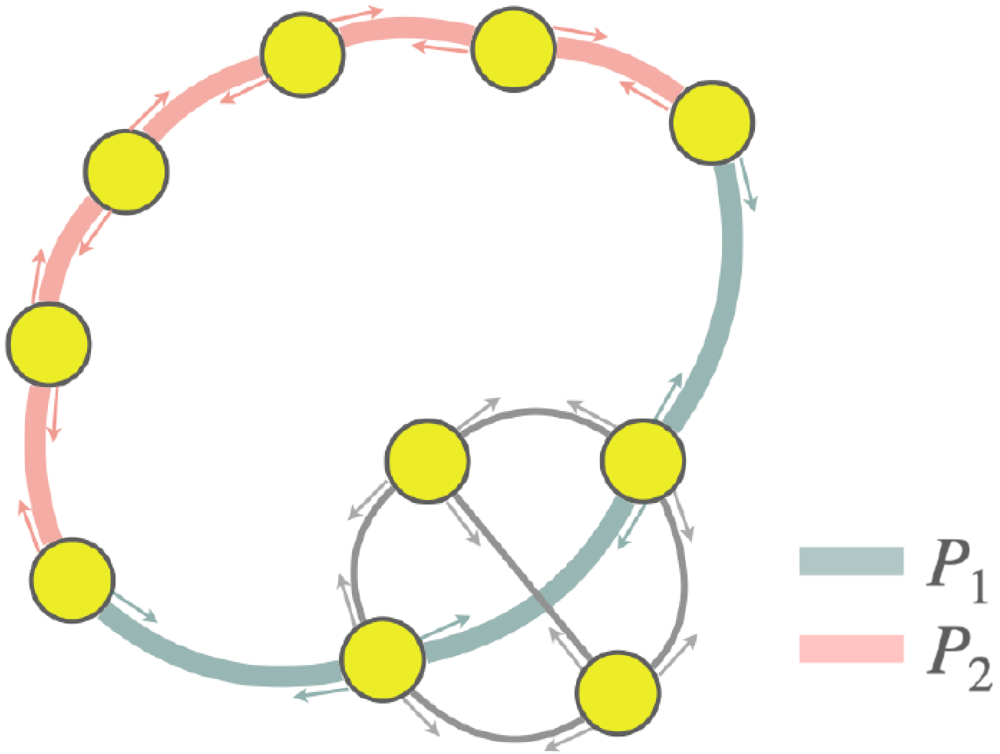
Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

"a routing/navigation process implies that communication flows from a specific source to a specific target along the fastest or most direct route, which implies global knowledge about the network topology."

"a diffusion process implies that communication occurs in the absence of specific targets, or that, even if targets are specified, a lack of knowledge about global network topology prevents particles or messages from taking shortest paths."



vs.



Goñi et al.: *PLoS ONE* **8** (2013) e58070

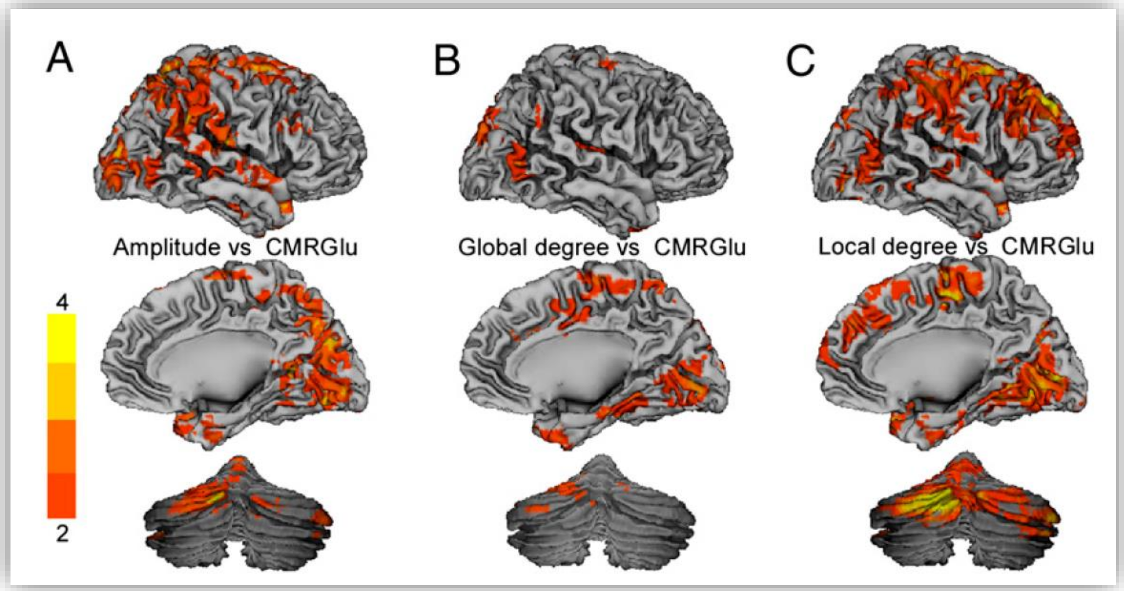
Affirmations like:

"The shortest path plays an important role in the information transmission of a brain network, and it is a very important measure to describe the internal structure of the brain network. The shortest path can transmit the information more quickly and reduce brain consumption."

Liu et al.: *Complexity* (2017), 8362741.

Contrasts with facts like:

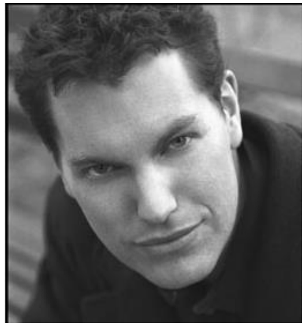
"A higher degree of connectivity was associated with nonlinear increases in metabolism."



Tomasi et al.: *Proc. Nat. Acad. Sci. USA* **110** (2013) 13642–13647.

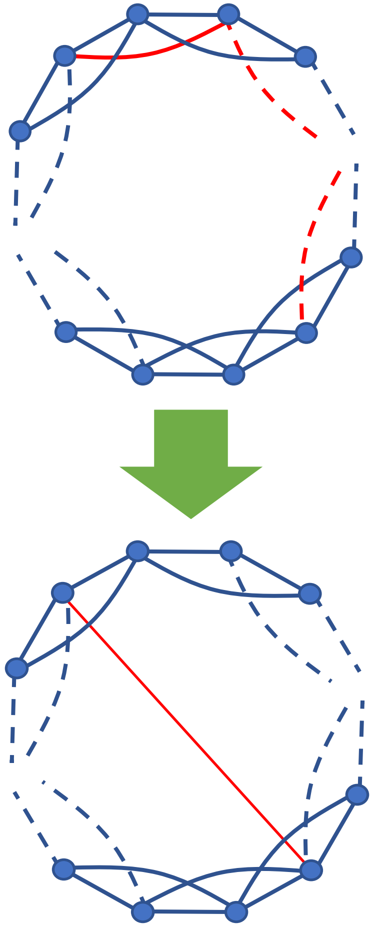
The idea of "shortest path" is implicit in the Watts-Strogatz and Barabási-Albert models

Collective dynamics of 'small-world' networks
***Nature* 393 (1998) 440-442**



Watts

Strogatz



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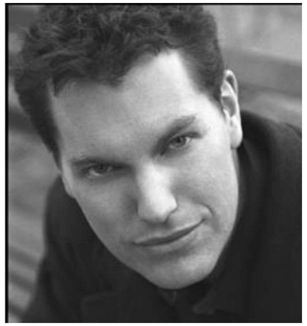
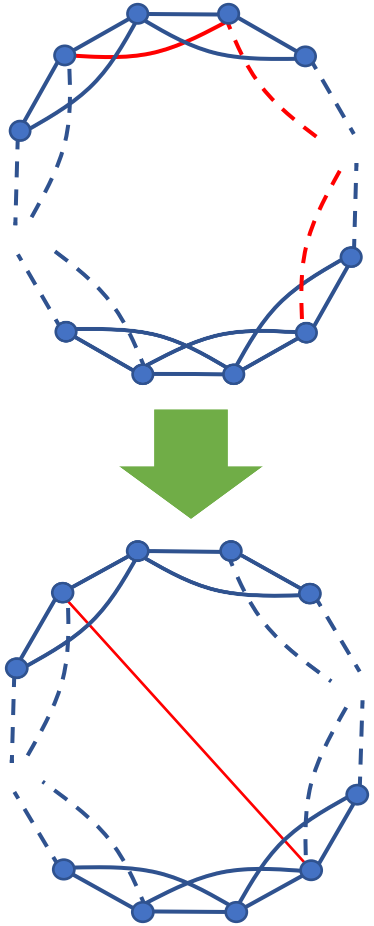


Watts

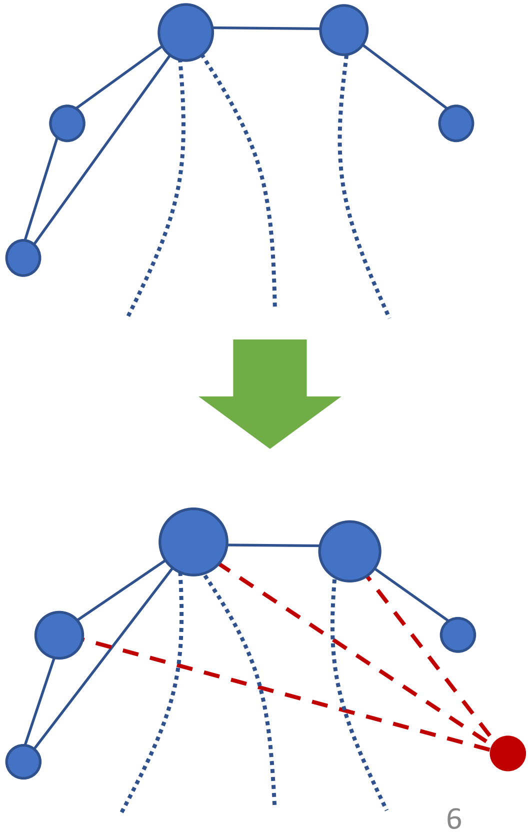
Strogatz



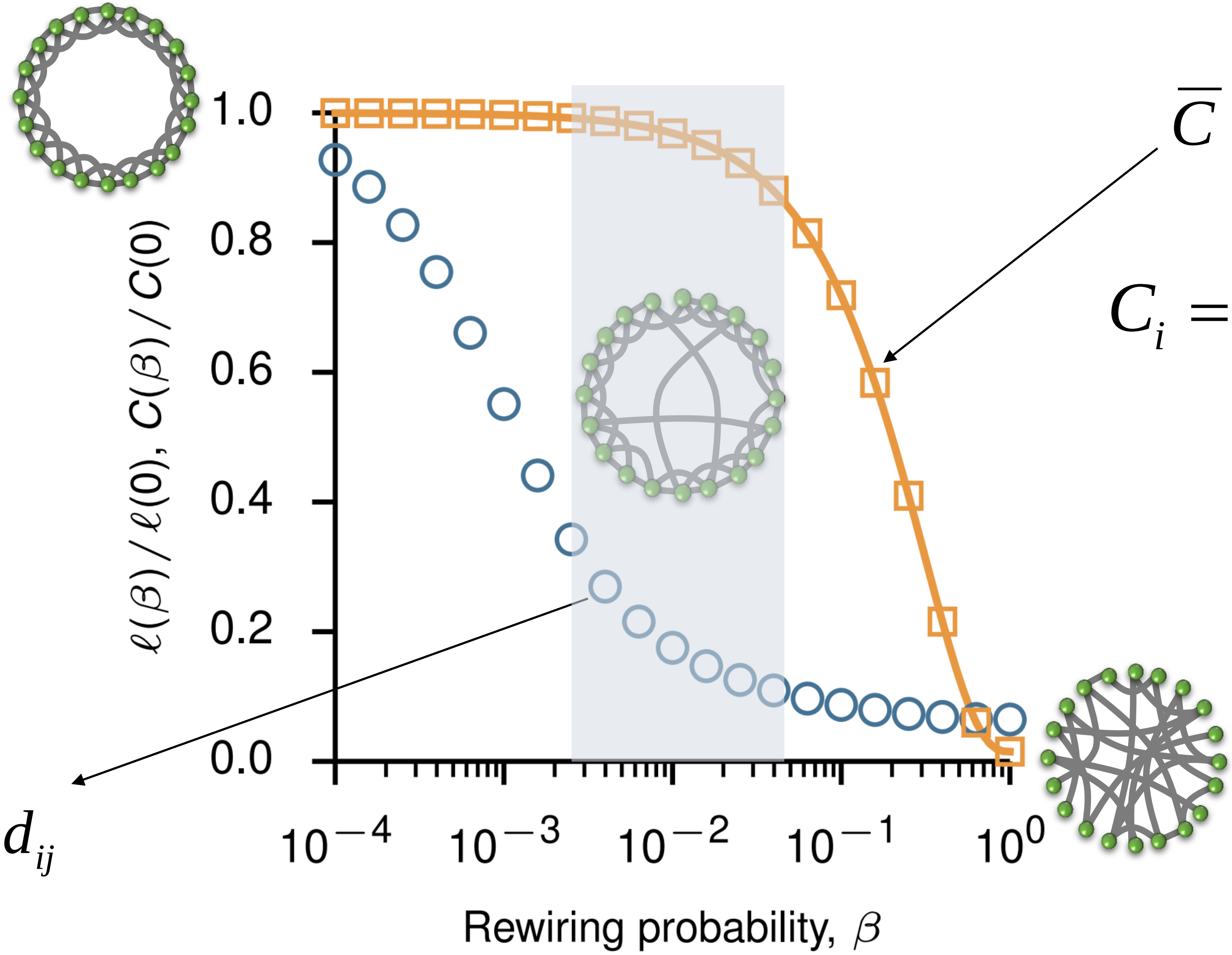
Emergence of Scaling in Random Networks
***Science* 286 (1999) 509-512**

Barabási

Albert



Watts-Strogatz model



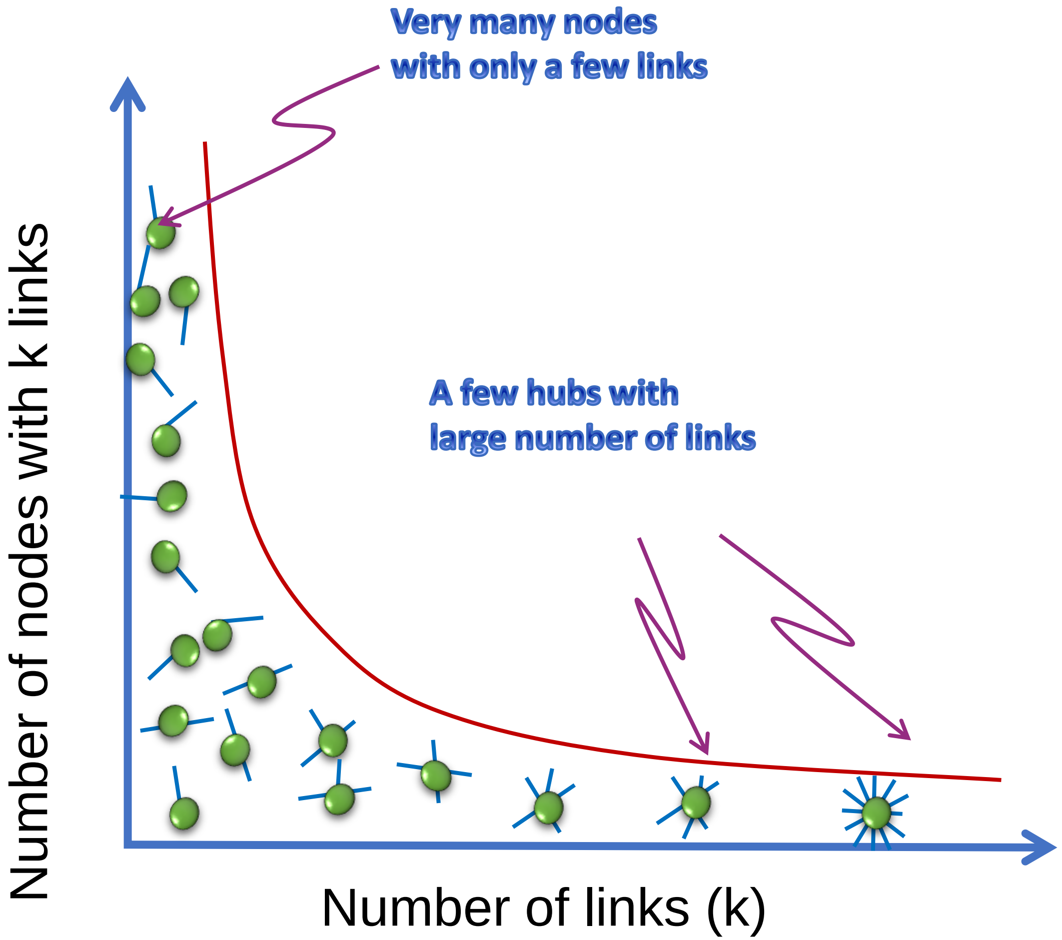
$$\bar{C} = \frac{1}{n}\sum_i C_i$$

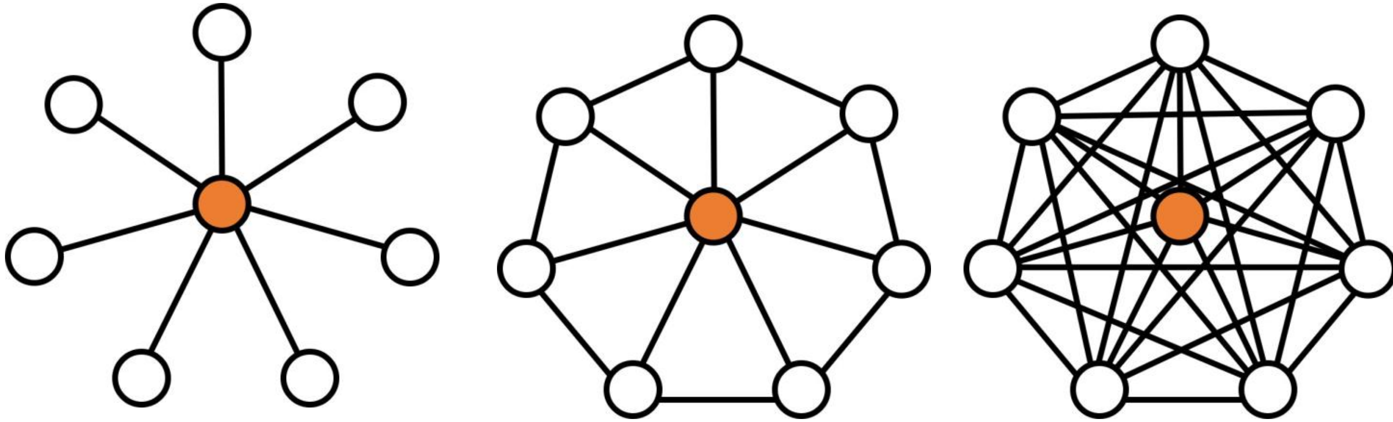
$$C_i = \frac{2t_i}{k_i\left(k_i - 1\right)}.$$

$$\bar{l} = \frac{1}{n(n-1)}\sum_{i,j} d_{ij}$$

Network models

Barabási-Albert model





The three ● nodes are equivalent in terms of the number of one-step paths connecting them to other nodes, but not in terms of the total number of routes connecting them to ○ nodes.

Preliminaries

Definition 1. *Let* $G=(V,E)$ *be a simple, undirected and connected graph. The matrix* $A=(A_{vw})_{n\times n}$ *is the adjacency matrix of G, and it has entries given by*

$$A_{vw}=\begin{cases}1 & \text{if } (v,w)\in E\\ 0 & \text{if } (v,w)\notin E.\end{cases}$$

Definition 2: *A walk of length l, is any sequence of (not necessarily different) nodes* $v_1,\ldots,v_l$ *such as for each i =1,…,l there is a link from* v_l *to* v_{l+1}*. It is closed if* $v_1 = v_l$ *.*

Lemma 3. *The number of walks of length l between the nodes v and w in a network is equal to:* $(A^l)_{vw}$.

Estrada: *The Structure of Complex Networks. Theory and Applications(OUP, 2011)*

Definition 4. *The communicability function is given by the total number of walks, weighted in decreasing order of their lengths, connecting the vertices p and q in a network G*

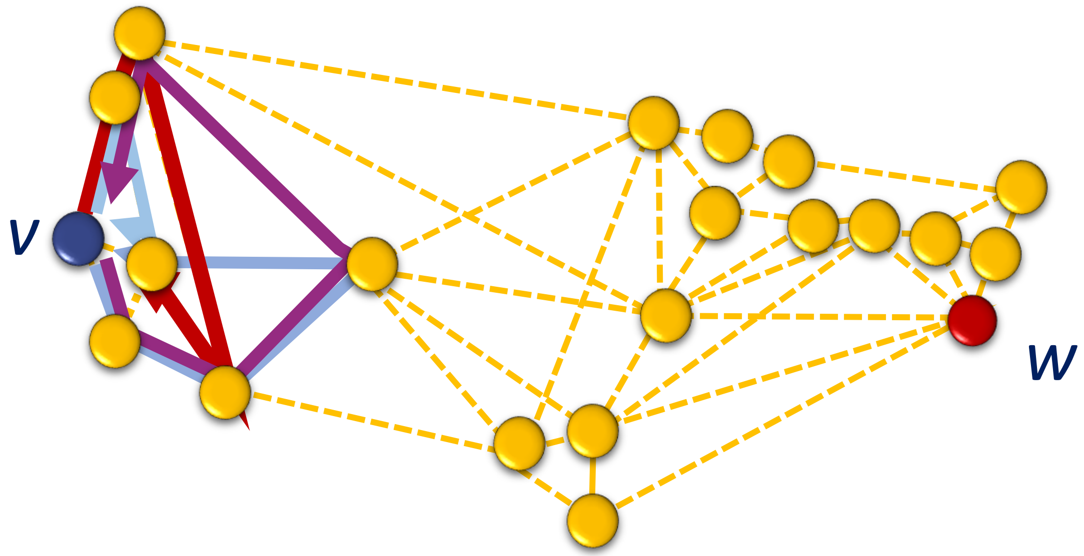
$$G_{pq}(\zeta) = \sum_{k=0}^{\infty} \frac{\left(\zeta^k A^k\right)_{pq}}{k!} = \left(e^{\zeta A}\right)_{pq}$$

where the weighting $(k!)^{-1}$ is selected among the several possibilities, and ζ is an empirical parameter.

Estrada & Hatano: *Phys. Rev. E.* ***77*** *(2008)* 036111.

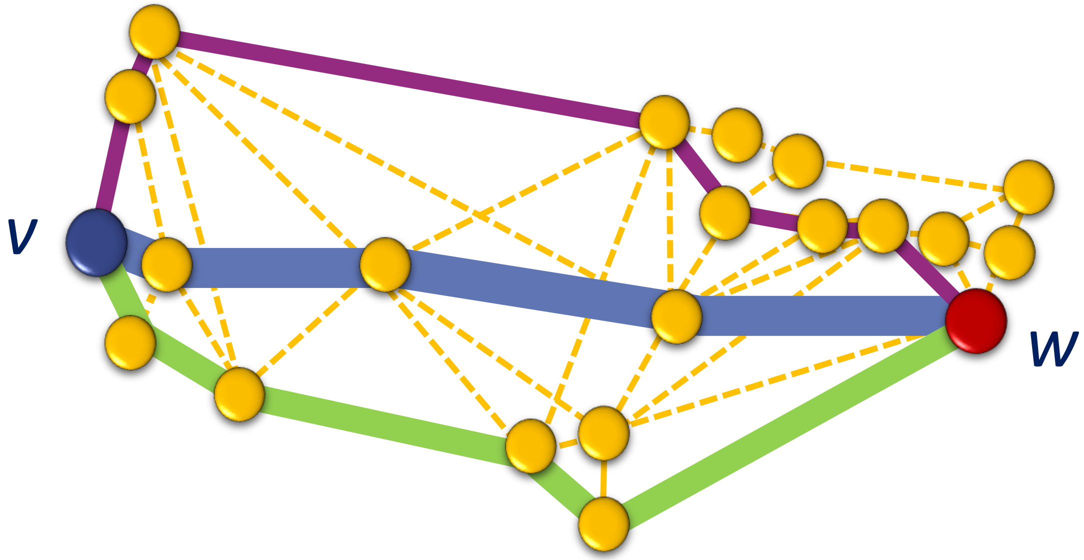
Preliminaries

$G_{vv}(\zeta)$: *accounts for all walks starting and ending at the same node*



It accounts for how much a node v retains an "item" at it, as the item goes back and forth to the node infinitely.

$G_{vw}(\zeta)$: *accounts for all walks starting at v and ending at w*



It accounts for how much a second node w attracts an "item" located at the node v, as the item ends up at w every time that it departs from v.

Communicability geometry

Definition 5. *The "resistance" that the network offers to an "item" at node v to go to the node w is given by the difference:*

$$R_{vw}(\zeta) := G_{vv}(\zeta) - G_{vw}(\zeta),$$

which is small when there are more walks "conductive" between the two nodes than those "regressive" to the origin. Hereafter $\zeta \equiv 1$ *for the sake of simplicity.*

Theorem 6. *The inter-node resistive function*

$$\xi_{vw} := R_{vw} + R_{wv} = \sum_{j=1}^{n} e^{\lambda_j} \left(\psi_{jv} - \psi_{jw} \right)^2$$

where ψ_{jv} *is the v-th entry of the eigenvector associated with the j-th eigenvalue* λ_j *of A, is a squared Euclidean distance between the nodes v and w in the network and induces an embedding of the network in an n-dimensional Euclidean sphere.*

Estrada: *Lin. Algebra Appl.* **436** (2012) 4317-4328.
Estrada et al.: *Discr. Appl. Math.* **176** (2014) 53-77.

Communicability geometry

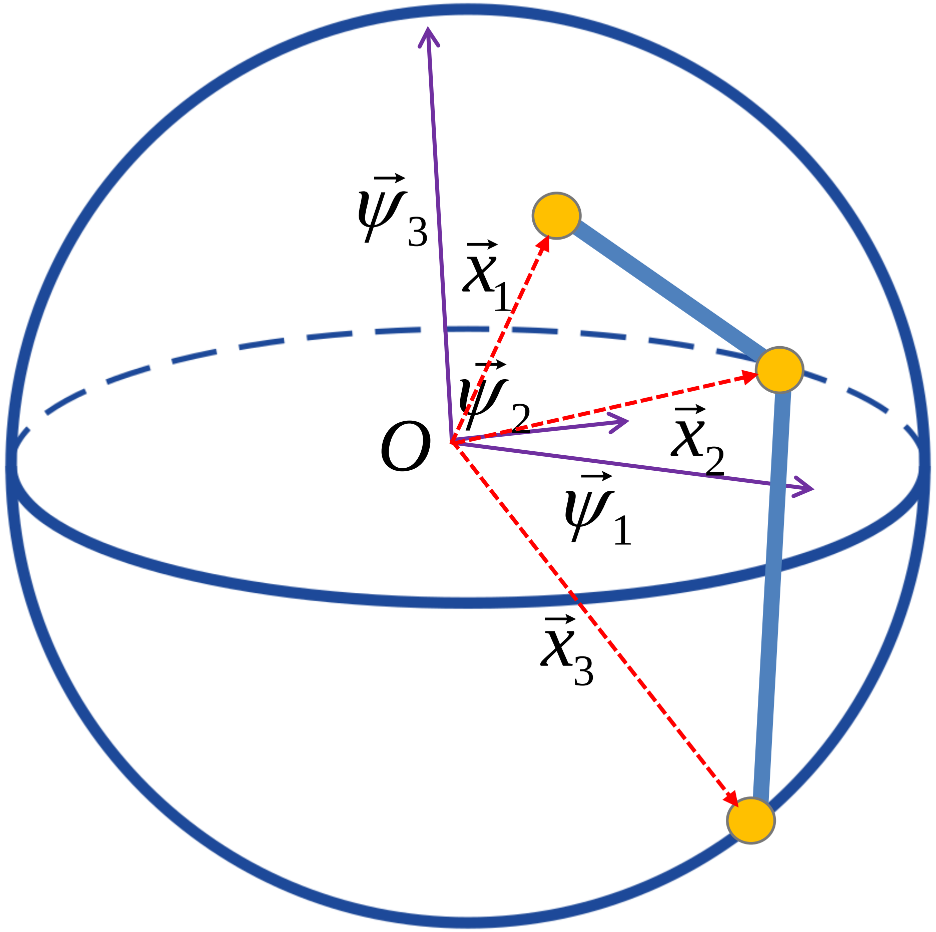
Theorem 7: *The communicability distance induces an embedding of a network into an Euclidean n-sphere*

$$S^n = \left\{ x \in \mathbb{R}^{n+1} : \|x\| = R \right\}$$

where:

$$R(G) = \frac{1}{2}\sqrt{\left(\vec{s}^T e^{-A} \vec{s} - \frac{\left(2 - \vec{s}^T e^{-A} \vec{1}\right)^2}{\vec{1}^T e^{-A} \vec{1}} \right)}$$

with $\vec{s} = \mathrm{diag}\left(e^{\gamma A}\right)$.



Estrada et al.: *Discrete Appl. Math.* **176** (2014) 53-77.

Communicability geometry

Definition 8: *A matrix* $D = (d_{ij})$ *is called Euclidean distance matrix (EDM) if there are points* $p^1, p^2, \dots, p^n$ *in some Euclidean space* $\mathbb{R}^r$ *such that*

$$d_{ij} = \left\| p^i - p^j \right\|^2$$

for all $i, j = 1, \dots, n$, *where* $\| \ \|$ *denotes Euclidean norm. If the points* $p^1, p^2, \dots, p^n$ *lie on a hypersphere, then the EDM is called speherical EDM or circum-Euclidean matrix.*

Therefore, $M = [\xi_{vw}]_{n \times n}$ *is a circum-Euclidean distance matrix.*

Communicability geometry

Lemma 9: M *is non singular and*

$$M^{-1} = -\frac{1}{2}e^{-A} + \frac{caa^T - (\varepsilon-2)b^T a^T - (\varepsilon-2)ab + db^T b}{2\left(cd - (\varepsilon-2)^2\right)}$$

where $a = e^{-A}\mathbf{1}$, $b = s^T e^{-A}$, $c = s^T e^{-A} s$, $d = \mathbf{1}^T e^{-A}\mathbf{1}$, $\varepsilon = s^T e^{-A}\mathbf{1}$ *and* $s = diag\left(e^A\right)$.

Proof:

- *Use the Sherman-Morrison-Woodbury formula;*
- *Show that* $cd - (\varepsilon-2)^2 \neq 0$;
- *It can also be shown that* $\mathbf{1}^T M^{-1}\mathbf{1} \neq 0$ *as required for circum-EDM.*

Estrada: *Lin. Algebra Appl*. **656** (2023) 198-209.

Communicability geometry

Lemma 10: *Let* $\mathcal{L} := e^{-A} - \frac{aa^T}{d}$. *Then,* $\mathcal{L}$ *is the unique Laplacian matrix associated with the matrix* M *of the graph.*

Proof:

- *Use a result of Balaji and Bapat about nonsingular spherical EDM and Laplacians to find that*

$$M^{-1} = -\frac{1}{2}\mathcal{L} + \frac{M^{-1}\mathbf{1}\mathbf{1}^T M^{-1}}{\mathbf{1}^T M^{-1}\mathbf{1}}$$

- *Find* $M^{-1}\mathbf{1}\mathbf{1}^T M^{-1}$.

see: Balaji & Bapat: *Lin. Algebra Appl*. **424** (2007) 108-117.
Estrada: *Lin. Algebra Appl*. **656** (2023) 198-209.

Communicability geometry

Theorem 11: *Let* $\mathcal{L}$ *be the Laplacian matrix associated with the matrix* M *of the graph. Then,*

$$M = \rho\mathbf{1}^T + \mathbf{1}\rho^T - 2\mathcal{L}^+$$

where

$$\mathcal{L}^+ = e^A + \left(\frac{\mathbf{1}^T e^A \mathbf{1}}{n^2}\right)\mathbf{1}\mathbf{1}^T - \frac{1}{n}\left(\mathbf{1}\mathbf{1}^T e^A + e^A \mathbf{1}\mathbf{1}^T\right)$$

is the Moore-Penrose pseudoinverse of $\mathcal{L}$*, and* $\rho = diag\left(\mathcal{L}^+\right)$.

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Communicability geometry

Proof: *We use the fact that*

$$\mathcal{L}^{+} = \left(\mathcal{L} + \frac{1}{n}\mathbf{1}\mathbf{1}^{T} \right)^{-1} - \frac{1}{n}\mathbf{1}\mathbf{1}^{T}.$$

Let us write

$$\left(\mathcal{L} + \frac{1}{n}\mathbf{1}\mathbf{1}^{T} \right)^{-1} = \left(e^{-A} + \frac{1}{n}\mathbf{1}\mathbf{1}^{T} - \frac{aa^{T}}{d} \right)^{-1},$$

and apply twice the Sherman-Morrison-Woodbury formula to obtain

$$\left(\mathcal{L} + \frac{1}{n}\mathbf{1}\mathbf{1}^{T} \right)^{-1} = e^{-A} - \frac{ff^{T}}{n+g},$$

where $f = e^{A}\mathbf{1}$ *and* $g = \mathbf{1}^{T} e^{A}\mathbf{1}$.

Estrada: *Lin. Algebra Appl*. **656** (2023) 198-209.

Communicability geometry

We then obtain

$$\left(\mathcal{L}+\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)^{-1}=e^A+\left(\frac{n+g}{n^2}\right)\mathbf{1}\mathbf{1}^T-\frac{1}{n}\left(\mathbf{1}f^T+f\mathbf{1}^T\right).$$

Therefore,

$$\left(\mathcal{L}+\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)^{-1}-\frac{1}{n}\mathbf{1}\mathbf{1}^T=e^A+\left(\frac{g}{n^2}\right)\mathbf{1}\mathbf{1}^T-\frac{1}{n}\left(\mathbf{1}\mathbf{1}^Te^A+e^A\mathbf{1}\mathbf{1}^T\right).$$

We now prove that $M=\rho\mathbf{1}^T+\mathbf{1}\rho^T-2\mathcal{L}^+$. *First, let* $T_i:=\sum_{j=1}^{n}\left(e^A\right)_{ij}$ *be the total communicability of node i. Then,*

$$\mathcal{L}^+_{ij}=\left(e^A\right)_{ij}+\frac{g}{n^2}-\frac{1}{n}\left(T_i+T_j\right).$$

For total communicability see: Benzi & Klymko: *J. Complex Netw*. **1** (2013) 124-149.
Estrada: *Lin. Algebra Appl*. **656** (2023) 198-209.

Communicability geometry

Thus .

$$\mathcal{L}_{ii}^{+}+\mathcal{L}_{jj}^{+}-2\mathcal{L}_{ij}^{+}=\left(e^{A}\right)_{ii}+\frac{g}{n^{2}}-\frac{2T_{i}}{n}+\left(e^{A}\right)_{jj}+\frac{g}{n^{2}}-\frac{2T_{j}}{n}-2\left(\left(e^{A}\right)_{ij}+\frac{g}{n^{2}}-\frac{T_{i}+T_{2}}{n}\right)$$

$$=\left(e^{A}\right)_{ii}+\left(e^{A}\right)_{jj}-2\left(e^{A}\right)_{ij}$$

$$=M_{ij}.$$

Estrada: *Lin. Algebra Appl*. **656** (2023) 198-209.

Communicability geometry

Definition 12: *The effective resistance between two nodes* $i, j \in \{1, \ldots, n\}$ *of a graph with Laplacian L is given by*

$$R_{ij} = \left(e_i - e_j\right)^T L^+ \left(e_i - e_j\right)$$

where L^+ *is the pseudoinverse of* L. *The effective resistance matrix* $R = \left[R_{ij}\right]$ *is defined as*

$$R = r\mathbf{1}\mathbf{1}^T + \mathbf{1}\mathbf{1}^T r - 2L^+$$

where $r = diag\left(L^+\right)$.

The square root of R_{ij} *is metric, and* R *is a Euclidean distance matrix, i.e., it has nonnegative elements, zero diagonal and it is negative semidefinite on* $\mathbf{1}^\perp$.

Communicability geometry

Remark 13. *The communicability distance between two nodes* $i, j \in \{1, \ldots, n\}$ *of a graph can be expressed as*

$$\xi_{ij} = \left(e_i - e_j\right)^T \mathcal{L}^+ \left(e_i - e_j\right)$$

and represents an effective resistance between them when the edges of the graph are weighted in a specified way.

Corollary 14. *Let* Ξ *be any nonsingular circum-EDM. Then,* Ξ *is a resistance EDM of the graph with the edges appropriately weighted.*

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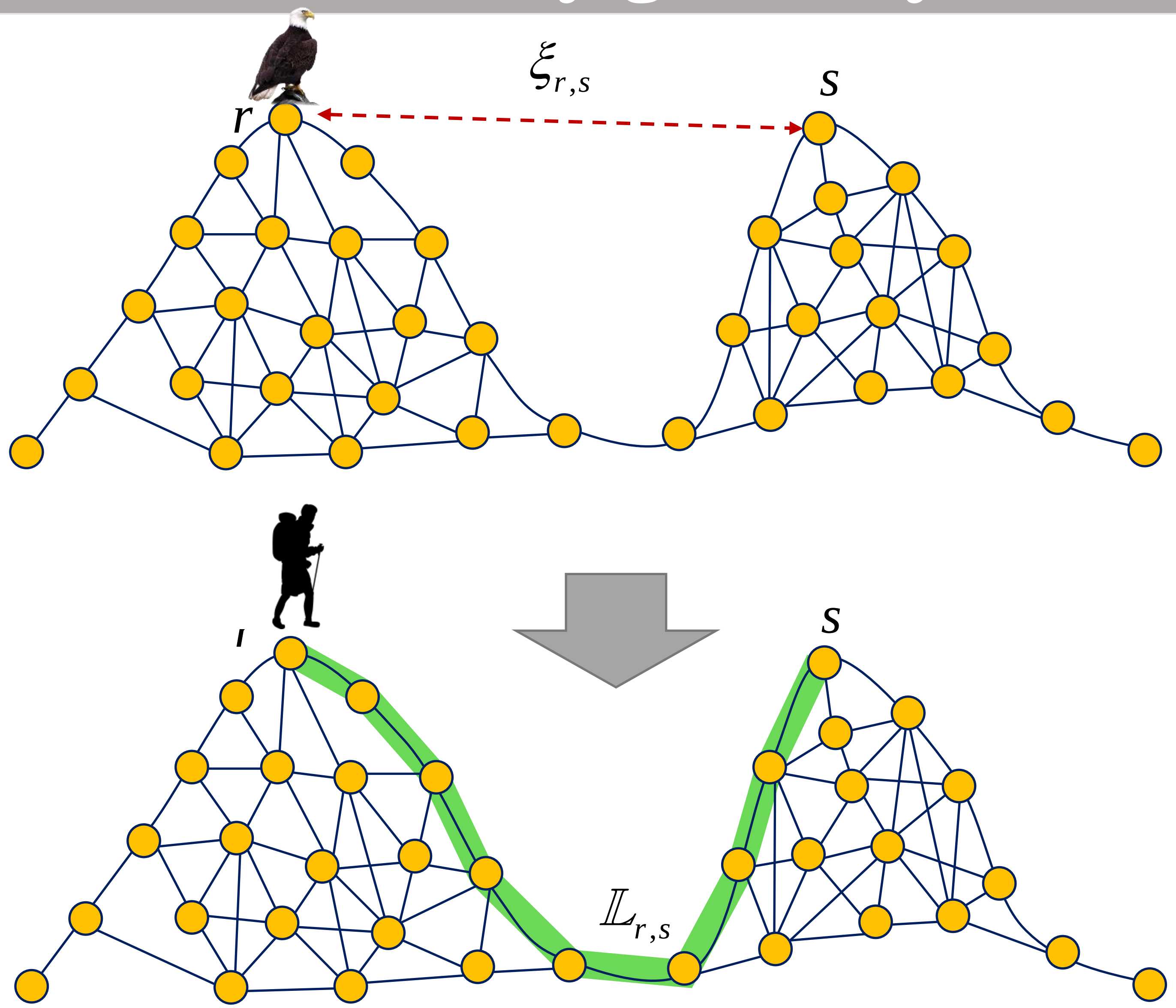
Communicability geometry



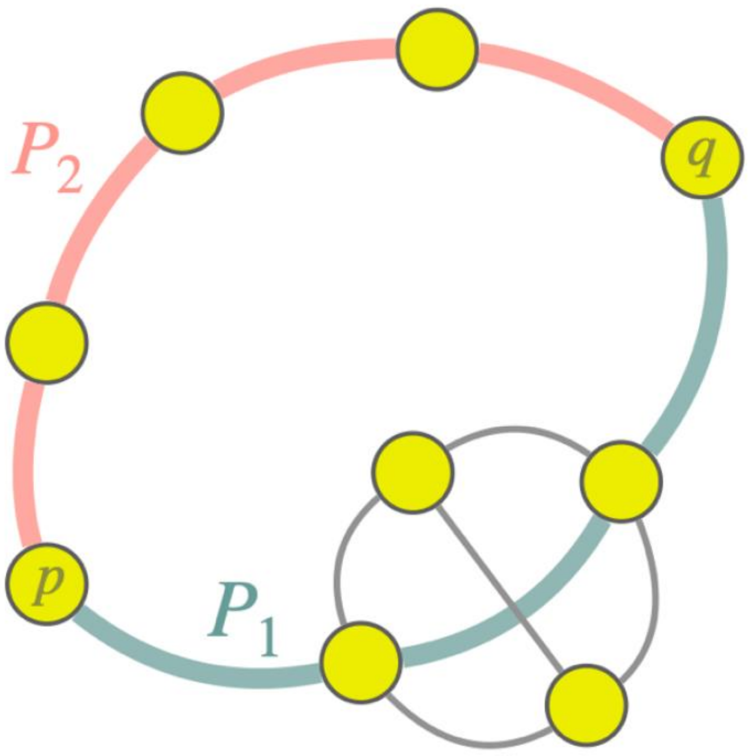
Geometrization: *consider every edge e=pq in E as a compact 1-dimensional manifold with boundary* $\partial e = p \cup q$*. Let the edge e=pq be given the metric* $\mathbb{L}_{p,q}$*, such that*

$$\tilde{e}_{pq} \underset{isom}{\cong} \begin{cases} \left[0, \xi_{pq}\right] & (p,q) \in E, \\ 0 & (p,q) \notin E. \end{cases}$$

Extend the distance metric $\mathbb{L}_{p,q}$ *on the edges of G via infima of lengths of curves in the geometrization of G. Then, the graph becomes metrically length spaces. This space is locally compact, complete and geodetic.*

Markvorsen: *Geom. Dedicata* **133** (2008) 7-34.
Bridson & Haefliger: *Metric Spaces of Non-positive Curvature*. Springer-Verlag, Berlin (1999). Chapter I.3. pp. 32-46

Definition 15. *Let $p(v \rightarrow w)$ be a path between the nodes v and w in G. The communicability resistive path length is defined as the sum of $\mathbb{L}_{p,q}$ for all edges in the path. Hereafter, $\mathbb{L}_{p,q}$ will be called resistive path length.*



Path lengths

topological	electric
$l(P_1) = 3$	$\mathbb{R}(P_1) = 2.15$
$l(P_2) = 4$	$\mathbb{R}(P_2) = 3.38$

Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Edge resistive length

Let $\mathbb{L}_{SP}(v,w)$ *be the resistive length of the topological SP. Let* $\mathbb{L}_{SRP}(v,w)$ *be the length of the shortest resistive path (SRP) between the same pair of nodes. Then,*

$$\mathbb{L}_{SRP}(v,w) \le \mathbb{L}_{SP}(v,w).$$

When, $\mathbb{L}_{SRP}(v,w) < \mathbb{L}_{SP}(v,w)$ *we call SRP a bypass between the nodes* v *and* w.

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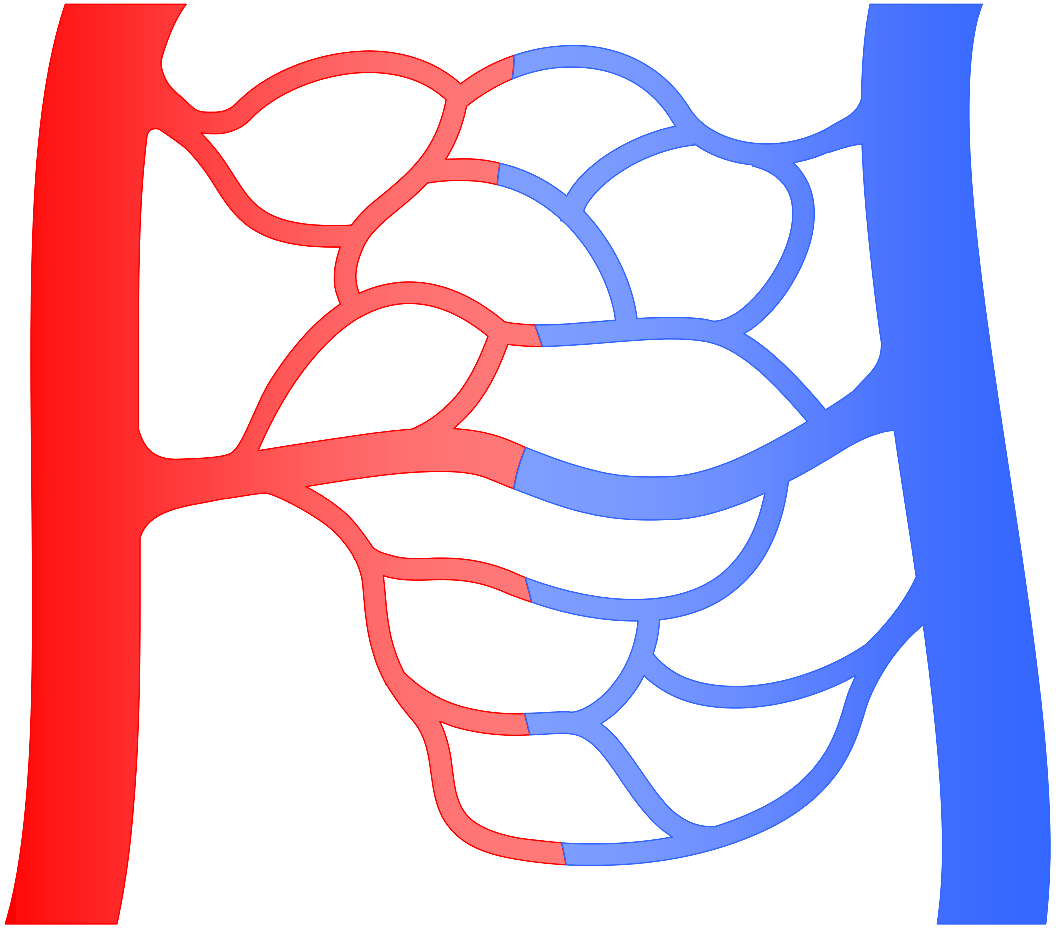
Bypasses in nature

"The collateral circulation is a network of specialized endogenous bypass vessels that is present in most tissues and provides protection against ischemic injury caused by ischemic stroke, coronary atherosclerosis, peripheral artery disease, and other conditions and diseases."



Faber et al.: *Arterioscler Thromb Vasc Biol*. 34 (2014) 1854-1859.

Edge resistive length

Let $\bar{\mathbb{L}}_{SP}(e)$ *be the mean edge resistive length along the SP, and let* $\bar{\mathbb{L}}_{SRP}(e)$ *be the same along the SRP. Obviously,*

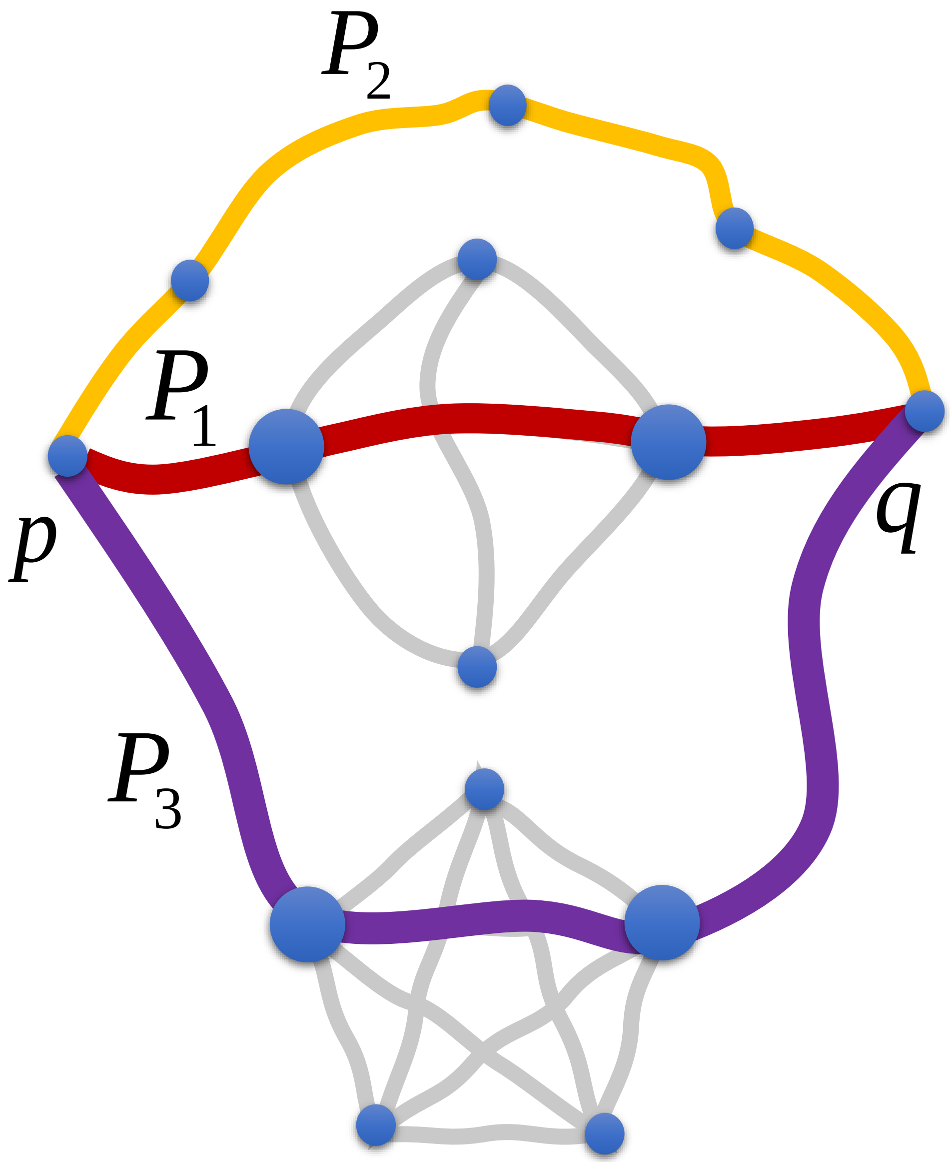
$$\bar{\mathbb{L}}_{SRP}(e)\cdot l_{SRP} \leq \bar{\mathbb{L}}_{SP}(e)\cdot l_{SP}.$$

Therefore,

$$\tilde{\varepsilon} := \frac{l_{SP}}{l_{SRP}} \geq \frac{\bar{\mathbb{L}}_{SRP}(e)}{\bar{\mathbb{L}}_{SP}(e)}$$

is an upper bound to the ratio of the mean edge resistive length in the SRP relative to that of the SP.

Why to use the bound?

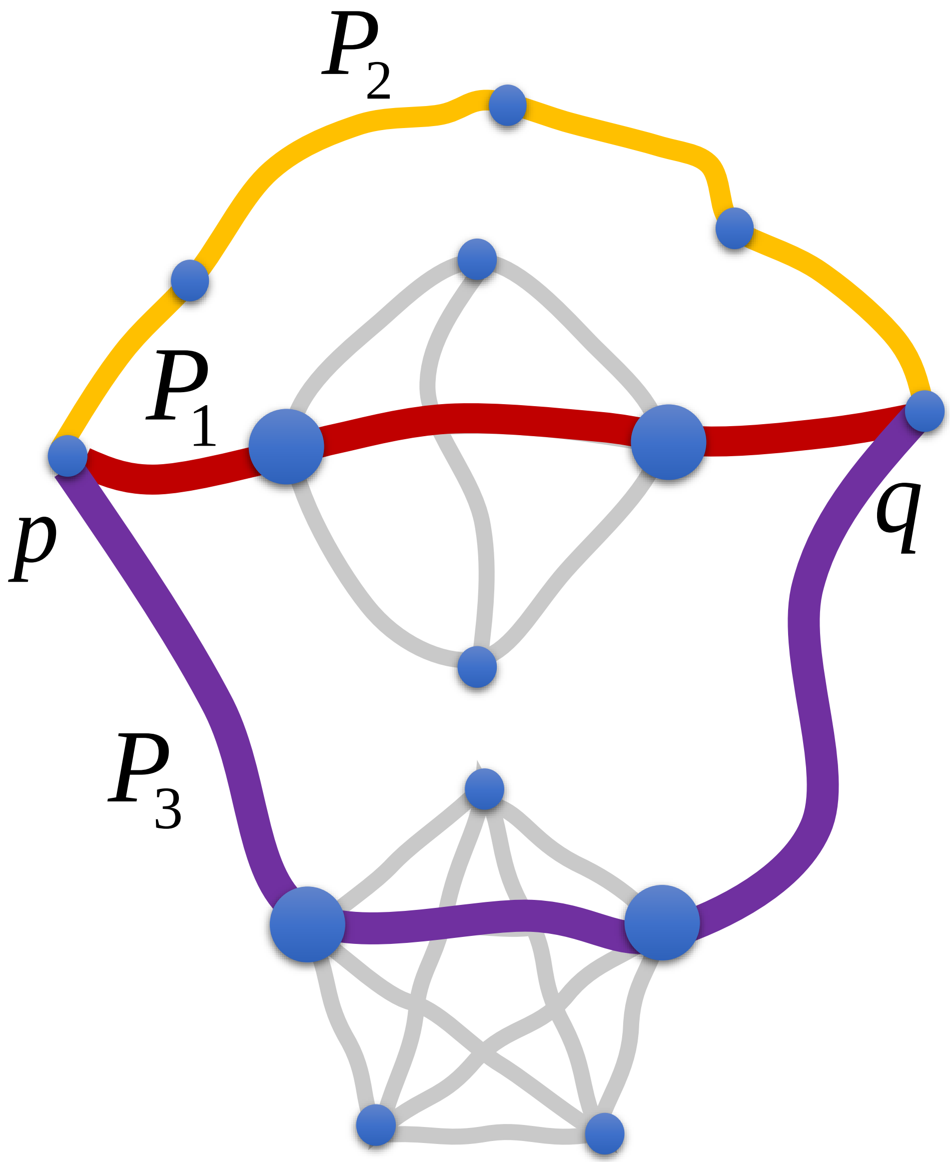


1) *Identify all shortest paths between the two nodes;*
2) *Identify the shortest resistive path;*
3) *Calculate* $\bar{\mathbb{L}}_{SRP}(e)/\bar{\mathbb{L}}_{SP}(e)$ *for every SP.*

$$\frac{\bar{\mathbb{L}}_{P_2}(e)}{\bar{\mathbb{L}}_{P_1}(e)} \approx 0.7444 \qquad \frac{\bar{\mathbb{L}}_{P_2}(e)}{\bar{\mathbb{L}}_{P_3}(e)} \approx 0.5739$$

Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Why to use the bound?



1) *Identify ONE shortest paths between the two nodes;*
2) *Identify THE shortest resistive path;*
3) *Calculate* l_{SP} / l_{SRP} :

$$3/4 = 0.75$$

Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Theoretical model in a nutshell

Definition 16. *Let*

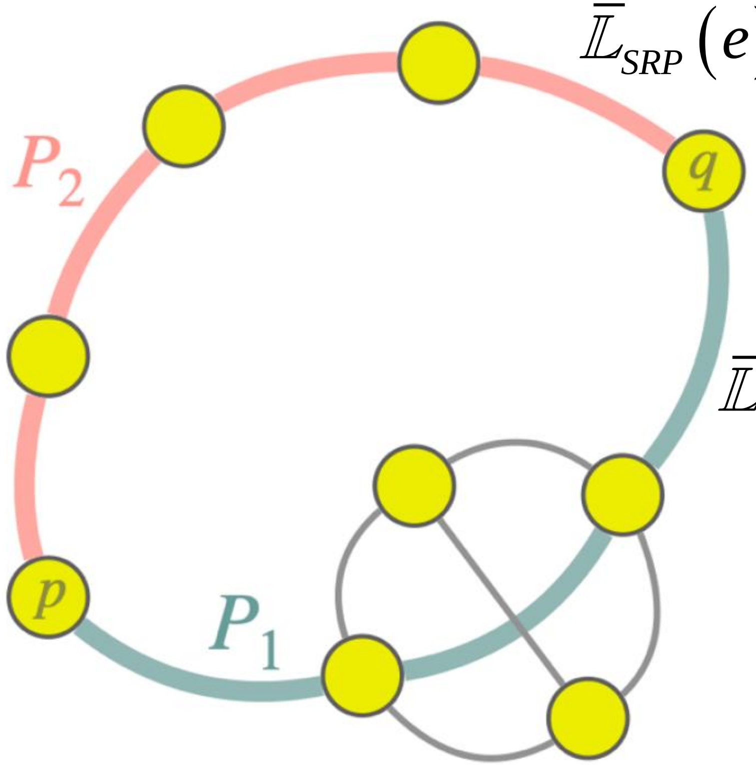
$$\varepsilon_{bf}(s,t) := 100\left(\frac{l_{SP}(s,t)}{l_{SRP}(s,t)} - 1\right)$$

be the least percentage of "energy" that an "item" traveling between the nodes s and t can save per edge as a result of selecting the SRP instead of the SP. The average of this quantity across all posible pairs of nodes in the network is

$$\varepsilon = \langle \varepsilon_{bf}(s,t) \rangle$$

Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Theoretical model in a nutshell



$\bar{\mathbb{L}}_{SRP}(e) \approx 1.200$

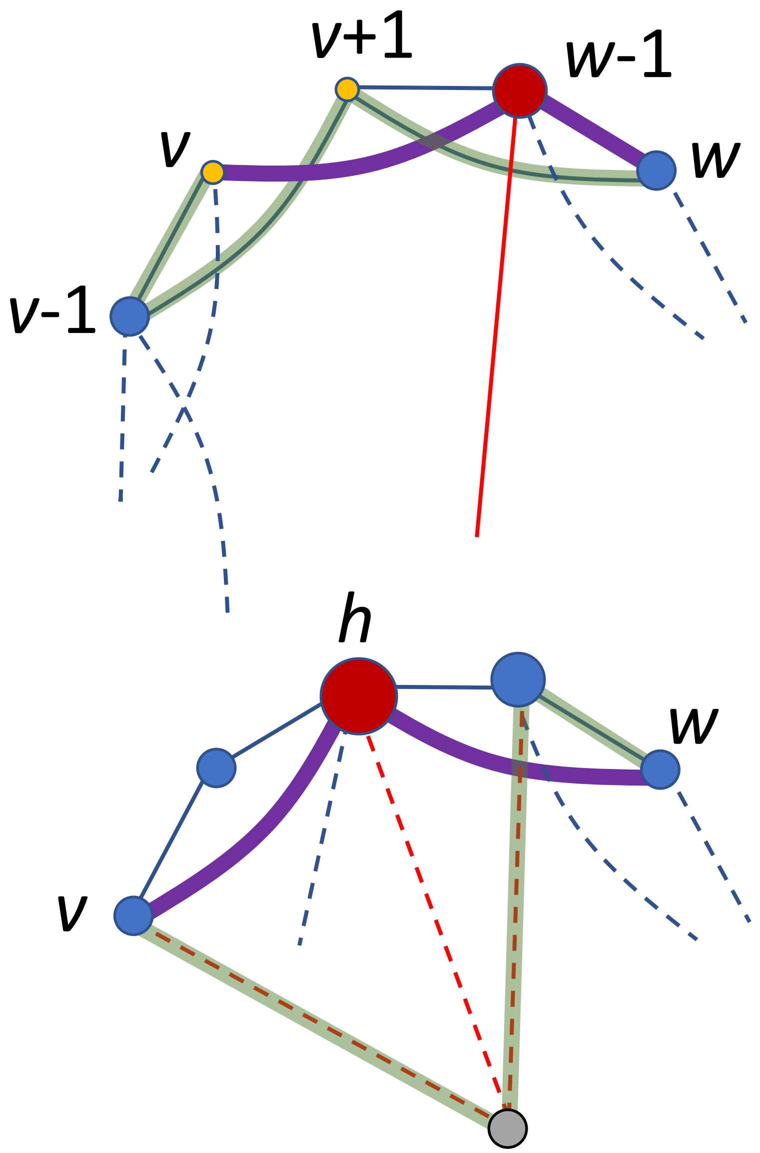
$\bar{\mathbb{L}}_{SP}(e) \approx 1.6967$

$$\frac{\bar{\mathbb{L}}_{SRP}(e)}{\bar{\mathbb{L}}_{SP}(e)} \approx 0.7073$$

$$\varepsilon = 100(3/4-1) = -25\%$$

Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Back to the models



$$q_{vw} = \frac{G_{vw}}{\sum_{i<j} G_{ij}}$$

all weighted walks between v and w

all weighted walks between every pair of nodes in the network

Walk Richness

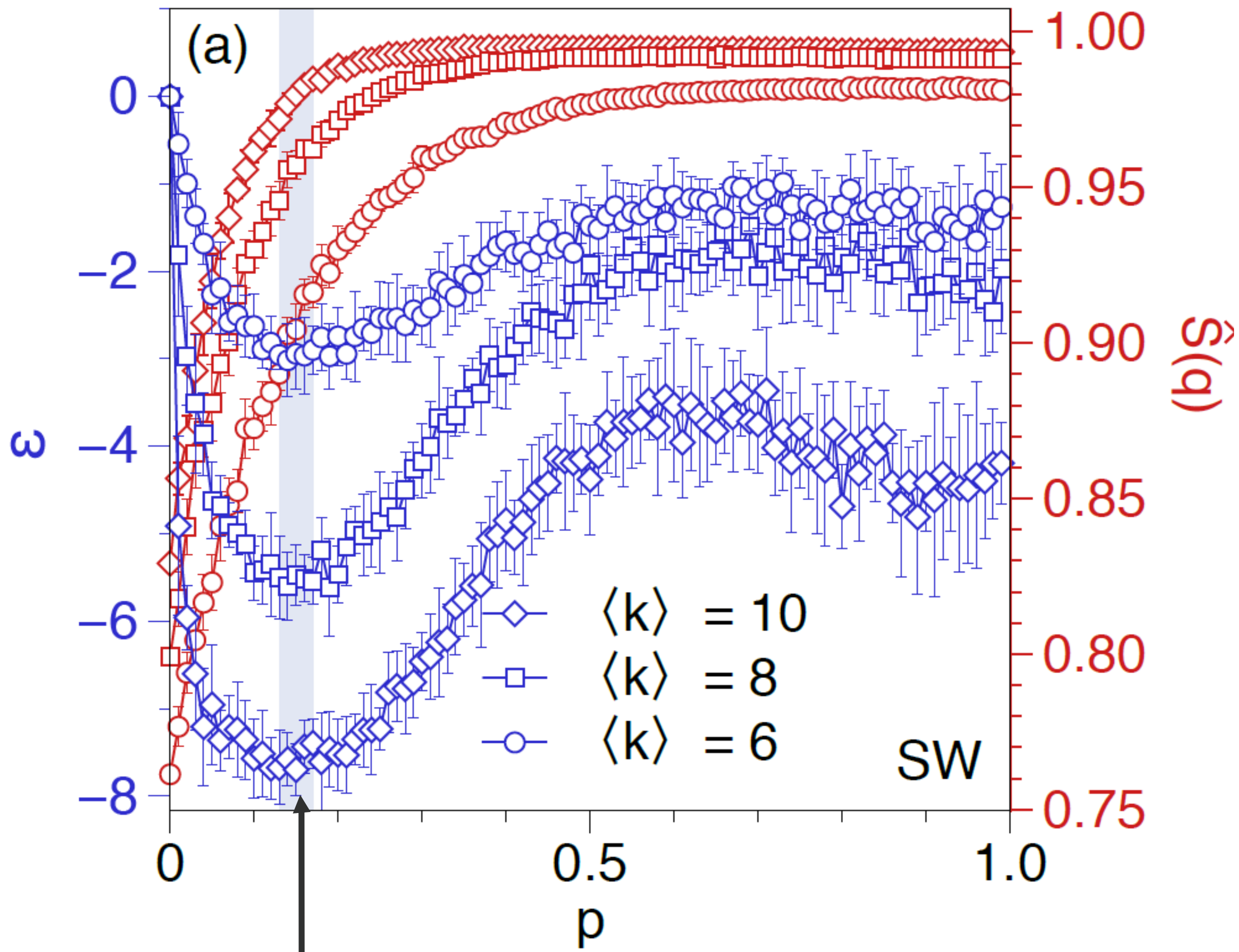
$$S(q) = -\frac{1}{2}\sum_{v<w} q_{vw} \ln q_{vw}$$

$$0 \le \hat{S}(q) = \frac{S(q)}{\ln(n(n-1)/2)} \le 1$$

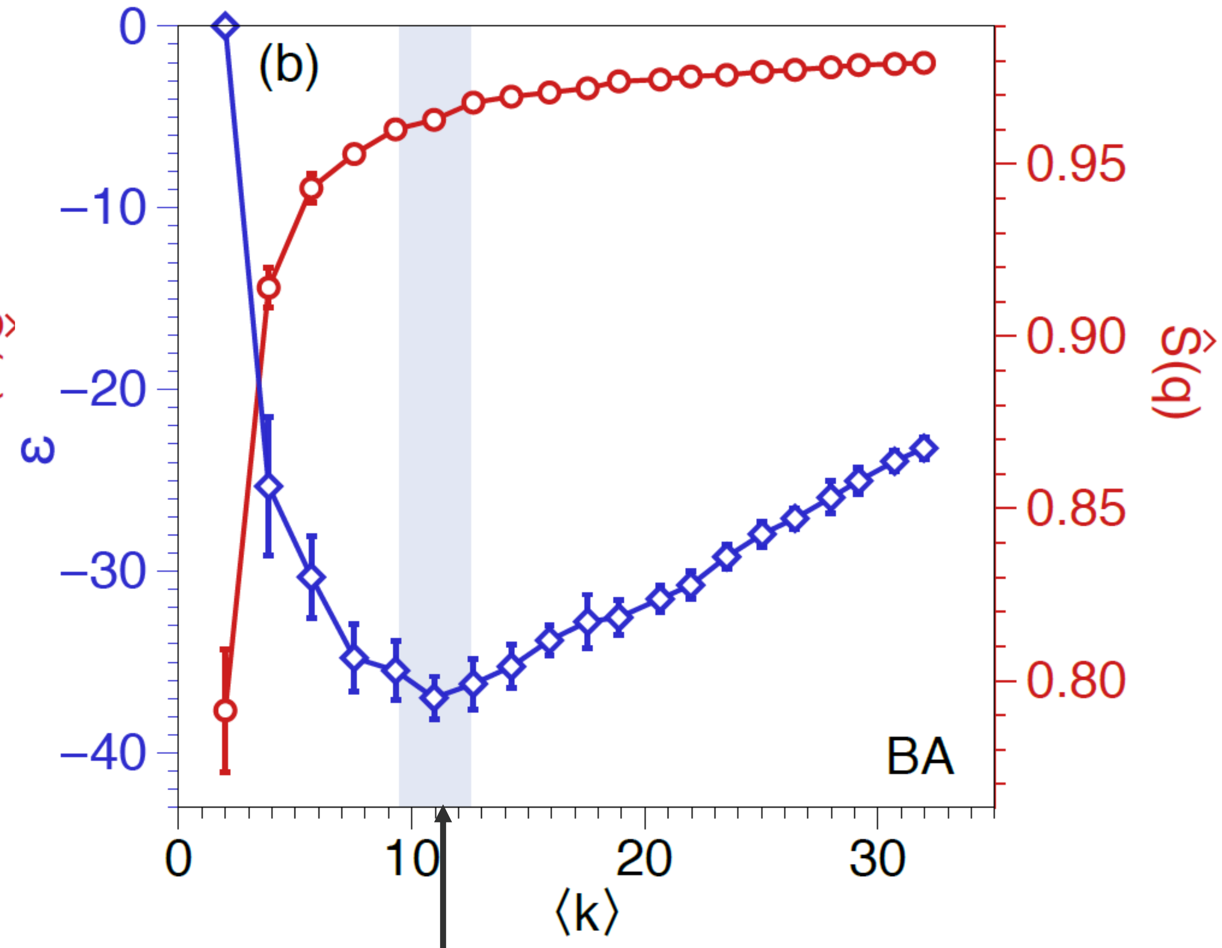
Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Back to the models

Watts-Strogatz model



Barabási-Albert model



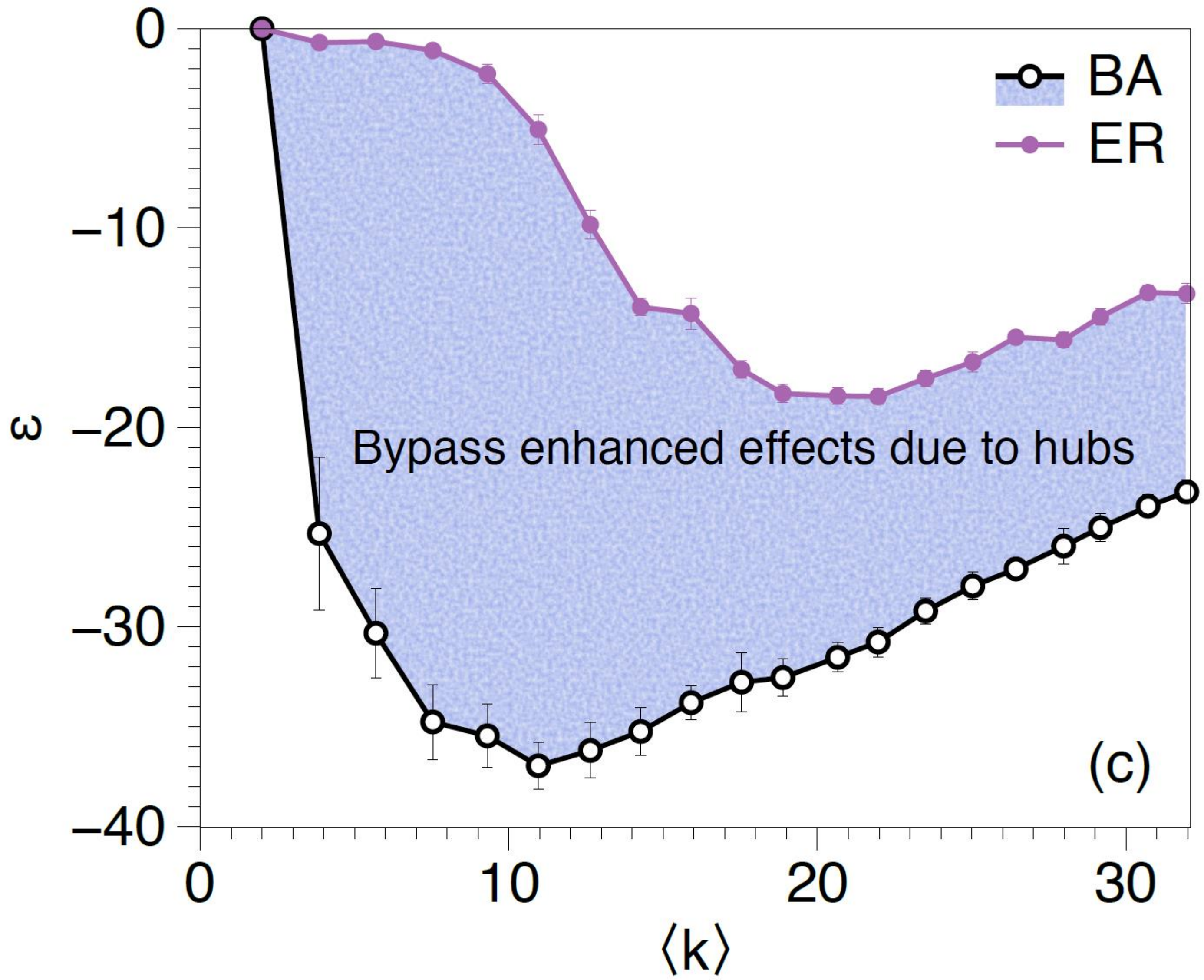
Good navigational point
(GNP)

In brief

- *Both processes (WS and BA) create enough walk richness as for certain bypasses among pairs of nodes EMERGE. Such bypasses avoid naturally the hubs of the network, allowing "ítems" navigating them to save "energy" in relation to what it will cost to navigate through the shortest paths.*

Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

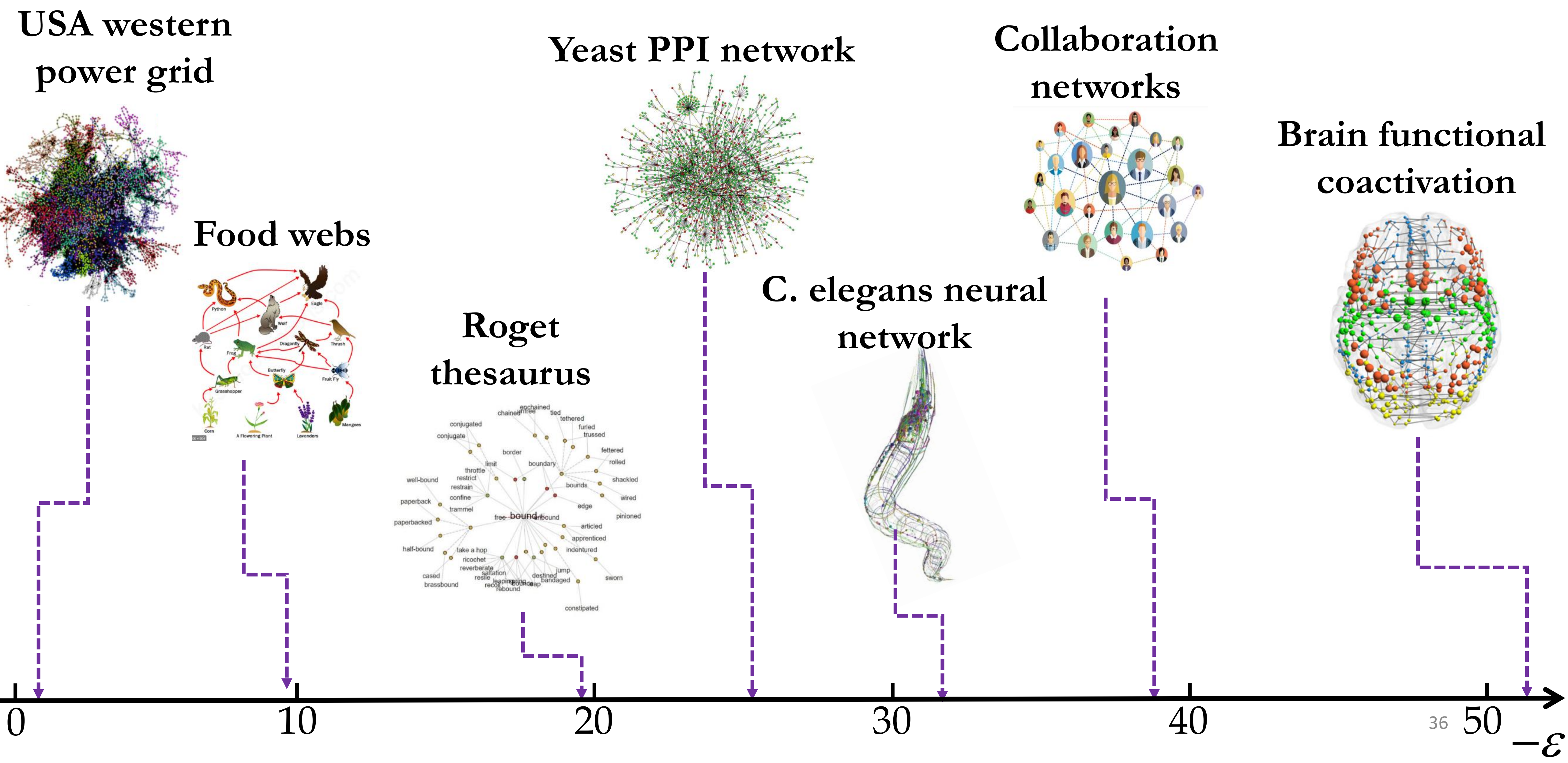
Back to the models



The hubs created by the PA mechanism enhance the emergence of efficient bypasses in relation to networks with more "normal-like" degree distributions.

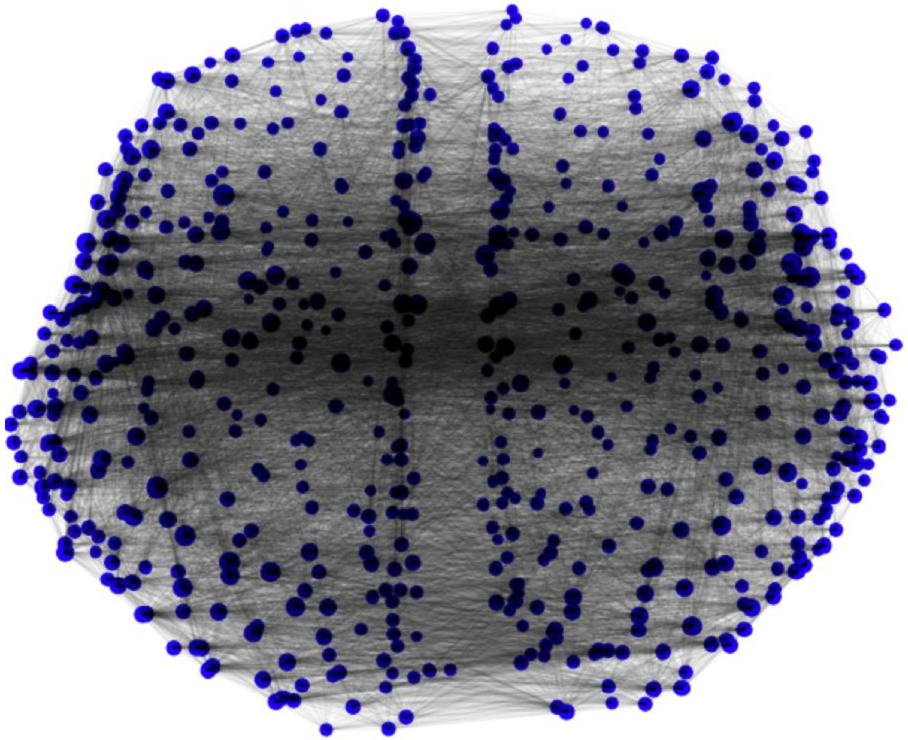
Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

What happen in the real-world?

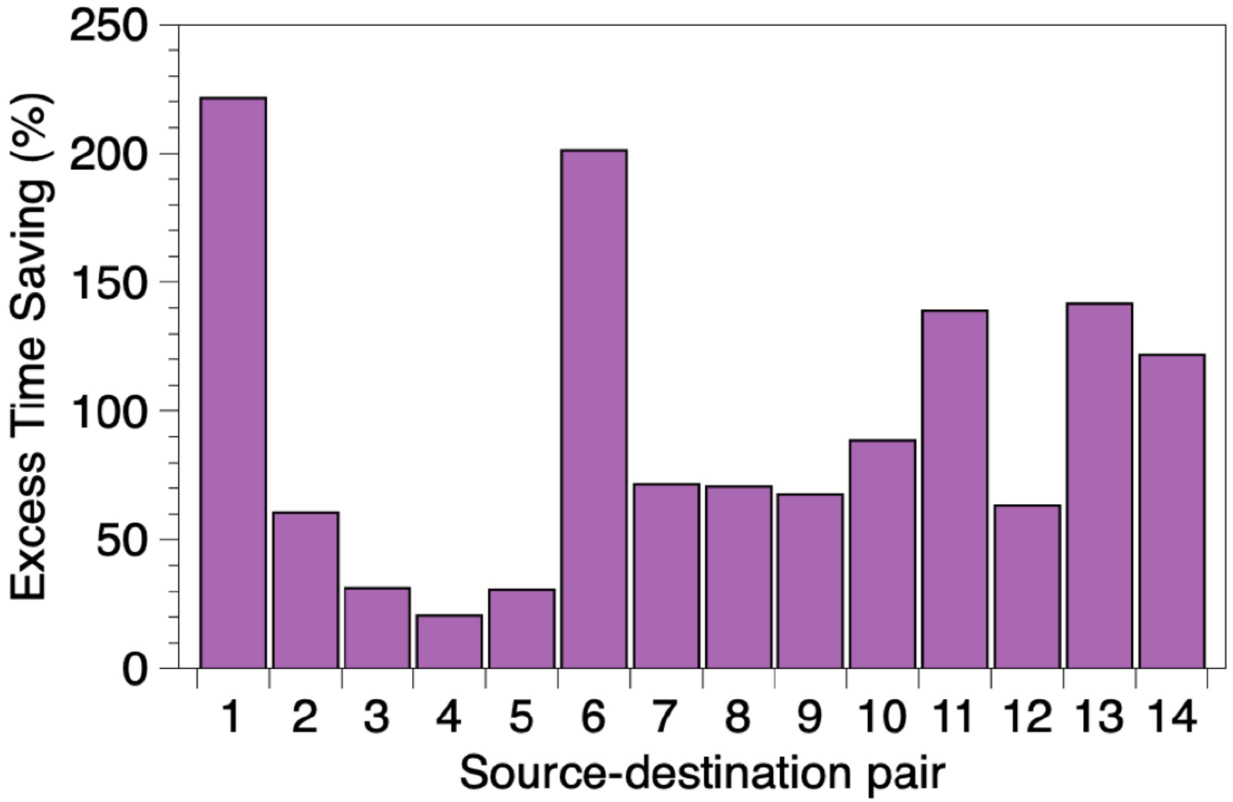


What happen in the real-world?

Brain functional coactivation



Time saving in diffusive processes

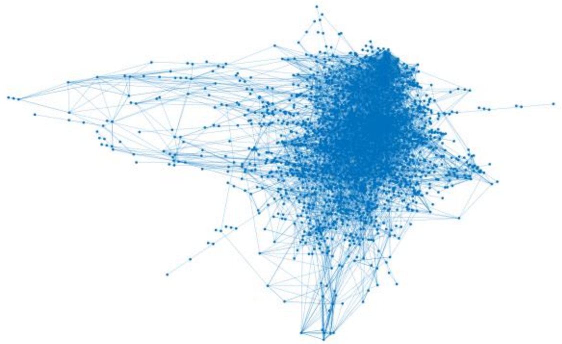


Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Dissimilarity between two networks

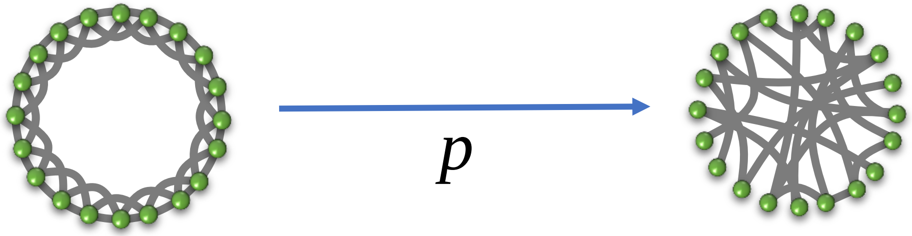
$$D(G,G') = \sqrt{\sum_{j=1}^{n} \left(\lambda_j(G) - \lambda_j(G') \right)^2}$$

Collaboration network QcGr



$n = 4158$

$m = 13,422$



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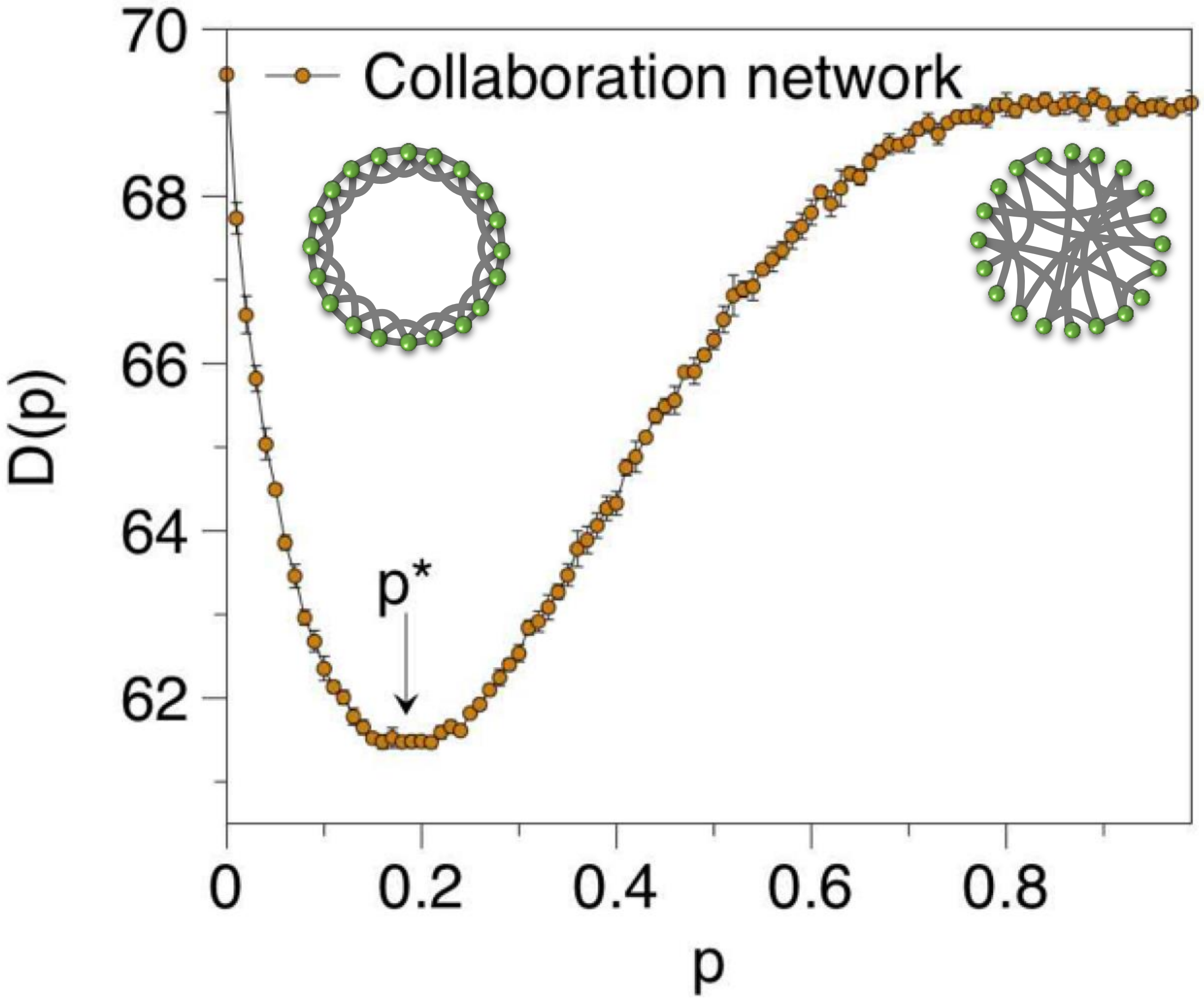
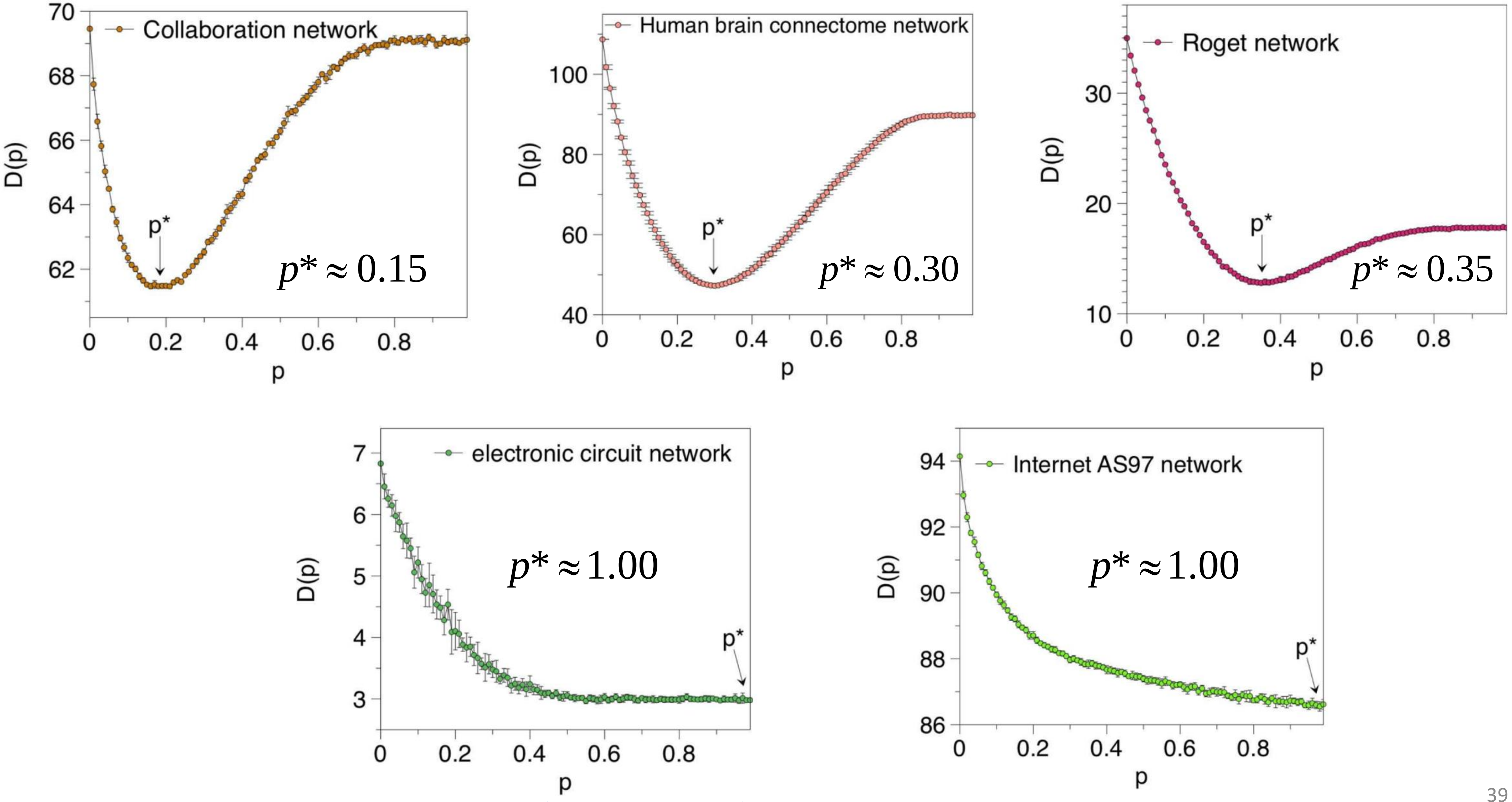


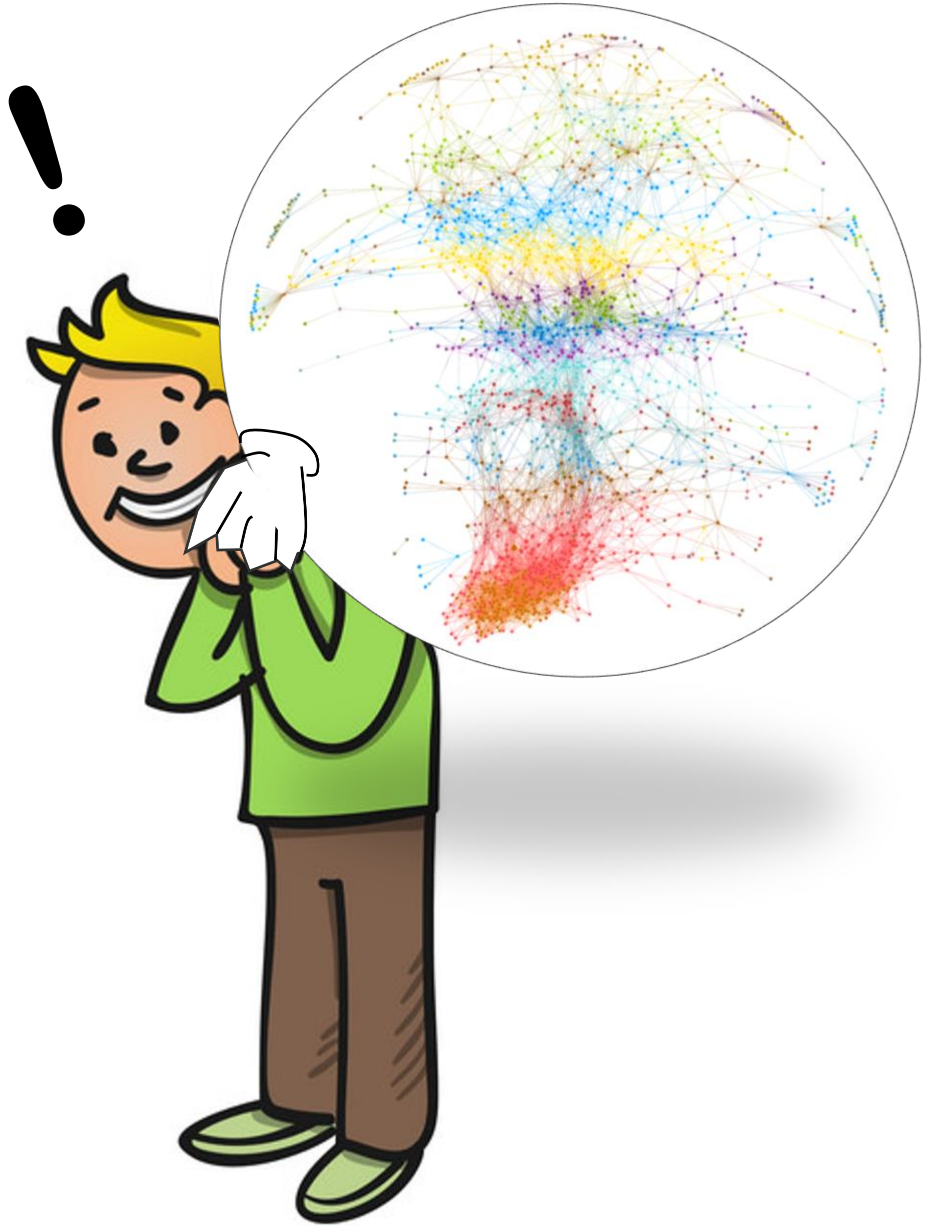
Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

Real-world networks and GNP



Estrada, Gomez-Gardeñes, Lacasa: *ArXiv*:2207.06813

What makes a network “complex”?



A network is said to be complex if it contains enough walk richness as for bypasses emerge as alternative best communication routes between pairs of nodes.

Simply put, bypasses seems to sustain complexity.

¡Muy próximamente!

Thank you!

How bypasses affect dynamics on networks?

Synchronization

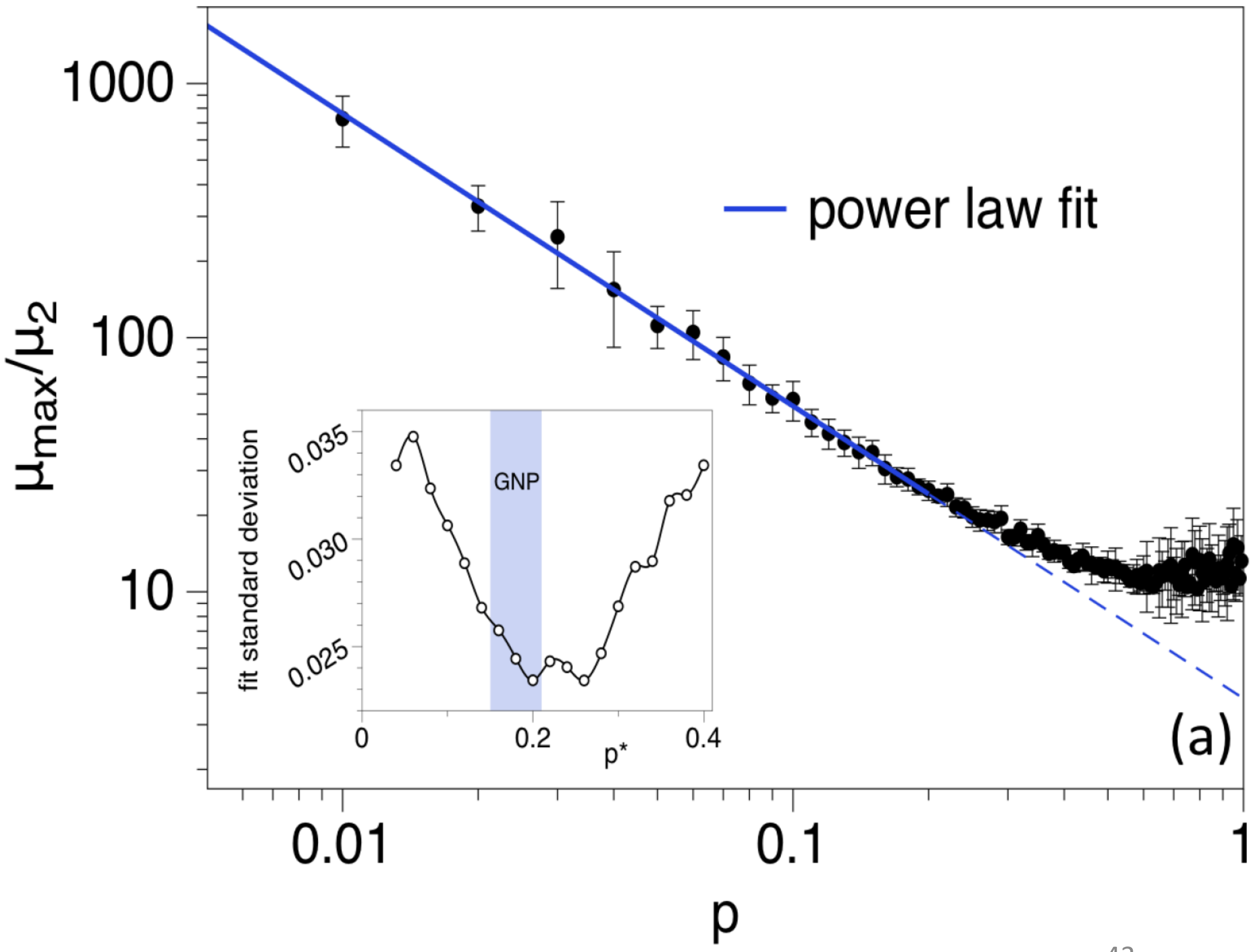
$$\dot{x}_i = F(x_i) - \sigma \sum_{j=1}^{n} L_{ij} H(x_j)$$

Synchronizability

Let $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_{\max}$ *be the eigenvalues of L. Network synchronizability is quantified by the ratio*

$$Q := \frac{\mu_{\max}}{\mu_2},$$

such that the smaller the eigenratio Q, the easier it is to synchronize the oscillators.



Barahona & Pecora: *Phys. Rev. Lett.* **89** (2002) 054101.

How bypasses affect dynamics on networks?

Epidemic spreading

Epidemic threshold

Let β *be the infection rate and* δ *be the curing rate for a disease spreading according to the model:*

$$\dot{p}_i(t) = \beta\left(1 - p_i(t)\right)\sum_{j\in\eta_i} A_{ij}\, p_j(t) - \delta\, p_i(t)$$

Then, infection will die out exponentially if the infection strength $\tau := \beta / \delta$ *satisfies:*

$$\tau \le \frac{1}{\lambda_1(A)}.$$