

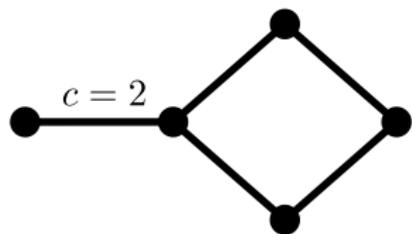
Matrix theory for Laplacian and effective resistance matrices

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VI Jornadas ALAMA: Laplacian and M-matrices in graphs
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MAX-PLANCK-INSTITUT
FÜR MATHEMATIK
IN DEN NATURWISSENSCHAFTEN

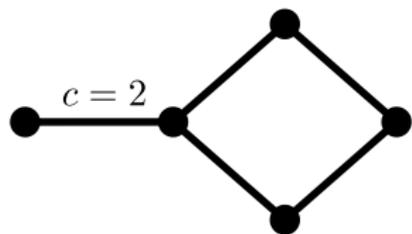




Graph $G = (V, E, c)$

$$L = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Laplacian matrix L



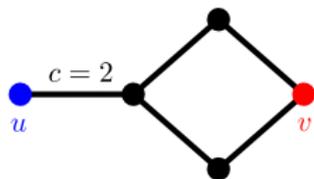
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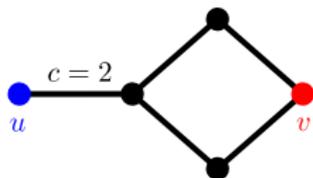
Laplacian matrix L

Definition (Laplacian matrix)

A Laplacian matrix L is a real symmetric matrix with $(L)_{ij} \leq 0$ for all $i \neq j$ and $\ker(L) = \text{span}(1)$.



$$u \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix} \xrightarrow{(\cdot)^{-1}} \frac{1}{4} \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 2 & 4 & 4 & 6 \end{pmatrix} v$$



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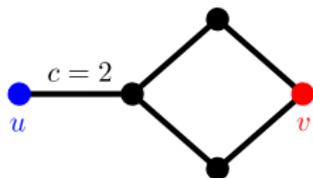
$$\Omega = \frac{1}{4} \begin{pmatrix} 0 & 2 & 5 & 5 & 6 \\ 2 & 0 & 3 & 3 & 4 \\ 5 & 3 & 0 & 4 & 3 \\ 5 & 3 & 4 & 0 & 3 \\ 6 & 4 & 3 & 3 & 0 \end{pmatrix} \begin{matrix} u \\ v \end{matrix}$$

Resistance matrix Ω

$$\Omega_{uv} = (L \setminus \{u\})_{vv}^{-1} = (L \setminus \{v\})_{uu}^{-1}$$

Definition (Resistance matrix)

The resistance matrix Ω of a graph has entries $(\Omega)_{uv} := (L \setminus \{u\})_{vv}^{-1}$ and zero diagonal.



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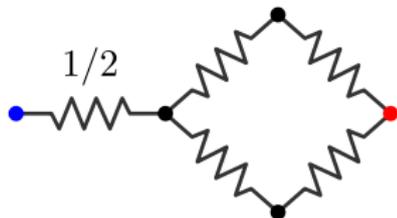
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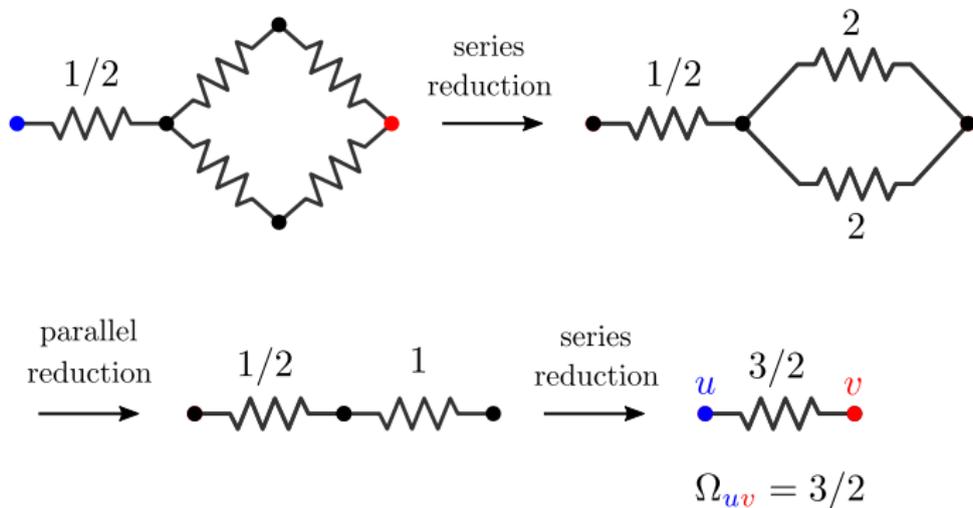
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(will also write $\omega_{uv} := \Omega_{uv}$)

Motivation: electrical circuits



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$$L = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Laplacian matrix L

$$\Omega = \frac{1}{4} \begin{pmatrix} 0 & 2 & 5 & 5 & 6 \\ 2 & 0 & 3 & 3 & 4 \\ 5 & 3 & 0 & 4 & 3 \\ 5 & 3 & 4 & 0 & 3 \\ 6 & 4 & 3 & 3 & 0 \end{pmatrix}$$

Resistance matrix Ω

- ▶ Framework for studying L and Ω
 - ▷ Schur complement
 - ▷ Fiedler–Bapat identity
- ▶ Applications of the framework
- ▶ Two invariants: p and σ^2

► Schur complement

Definition (Schur complement)

Let $M = \begin{pmatrix} M_{AA} & M_{AA^c} \\ M_{A^cA} & M_{A^cA^c} \end{pmatrix}$ with $M_{A^cA^c}$ invertible, then the *Schur complement* M/A^c is defined as

$$M/A^c := M_{AA} - M_{AA^c}(M_{A^cA^c})^{-1}M_{A^cA}.$$

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Example:

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \text{ and } A = \{1, 2\}$$

$$M/\{3\} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} 1^{-1} (2 \ 2) = \begin{pmatrix} -3 & -2 \\ -2 & 0 \end{pmatrix}$$

► Fiedler–Bapat identity

Theorem (Fiedler–Bapat identity)

The Laplacian and resistance matrix of a graph satisfy

$$\begin{pmatrix} 0 & 1^T \\ 1 & \Omega \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4\sigma^2 & -2p^T \\ -2p & L \end{pmatrix},$$

where $p = \frac{\Omega^{-1}1}{1^T\Omega^{-1}1}$ and $\sigma^2 = \frac{1}{2} \frac{1}{1^T\Omega^{-1}1}$.

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Example:

$$4 \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 5 & 5 & 6 \\ 1 & 2 & 0 & 3 & 3 & 4 \\ 1 & 5 & 3 & 0 & 4 & 3 \\ 1 & 5 & 3 & 4 & 0 & 3 \\ 1 & 6 & 4 & 3 & 3 & 0 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 1.75 & -1 & .5 & -.5 & -.5 & -.5 \\ -1 & 2 & -2 & 0 & 0 & 0 \\ .5 & -2 & 4 & -1 & -1 & 0 \\ -.5 & 0 & -1 & 2 & 0 & -1 \\ -.5 & 0 & -1 & 0 & 2 & -1 \\ -.5 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

► Combining the ingredients

Property

$$(M^{-1})_{AA} = (M/A^c)^{-1}$$

$$\begin{array}{ccc} & \text{Schur} & \\ & \longrightarrow & \\ M & & M/A^c \\ \text{inverse} \updownarrow & & \updownarrow \text{inverse} \\ N & \longrightarrow & N_{AA} \\ & \text{submatrix} & \end{array}$$

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$$\textit{inverse} \circ \textit{Schur} = \textit{submatrix} \circ \textit{inverse}$$

$$\begin{array}{ccccc}
 & \text{sub} & & \text{Schur} & \\
 L_{AA} & \longleftarrow & L & \longrightarrow & L/A^c \\
 & & & & \\
 \updownarrow & & \updownarrow \text{Fiedler} & & \updownarrow \\
 & & \text{-Bapat} & & \\
 \Omega/A^c & \longleftarrow & \Omega & \longrightarrow & \Omega_{AA} \\
 & \text{Schur} & & \text{sub} &
 \end{array}$$

► **Application 1: inverse resistance matrix**

$$-\frac{1}{2} \begin{pmatrix} 0 & 1^T \\ 1 & \Omega \end{pmatrix} = \begin{pmatrix} 4\sigma^2 & -2p^T \\ -2p & L \end{pmatrix}^{-1}$$

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Theorem (Bapat 2004)

The Laplacian and resistance matrix of a graph satisfy

$$\Omega^{-1} = -\frac{1}{2}L + \frac{pp^T}{2\sigma^2}$$

► Application 2: Kron reduction

$$-\frac{1}{2} \begin{pmatrix} 0 & 1^T & 1^T \\ 1 & \Omega_{AA} & \Omega_{AA^c} \\ 1 & \Omega_{A^cA} & \Omega_{A^cA^c} \end{pmatrix} = \begin{pmatrix} 4\sigma^2 & -2p_A^T & -2p_{A^c}^T \\ -2p_A & L_{AA} & L_{AA^c} \\ -2p_{A^c} & L_{A^cA} & L_{A^cA^c} \end{pmatrix}^{-1}$$

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Proposition (Folklore)

All Schur complements of a Laplacian matrix exist and are Laplacian. Similarly, all submatrices of a resistance matrix are again resistance matrices. Furthermore Ω_{AA} is the resistance matrix of L/A^c .

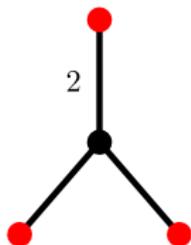
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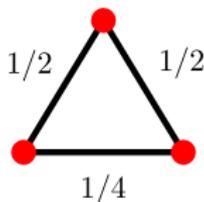
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Proof idea: show that $L_{a^c a^c} - L_{a^c a} L_{aa}^{-1} L_{aa^c}$ is Laplacian ($A^c = \{a\}$). The rest follows from Fiedler–Bapat + Schur complement



$$\begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -2 & -1 & -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}^T = \frac{1}{4} \left(\begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \right)$$



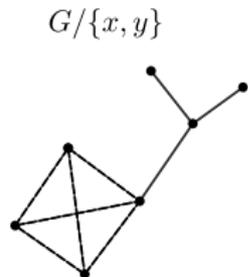
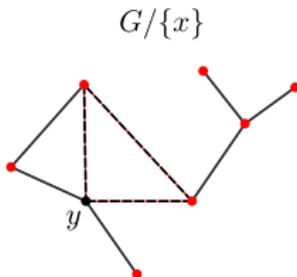
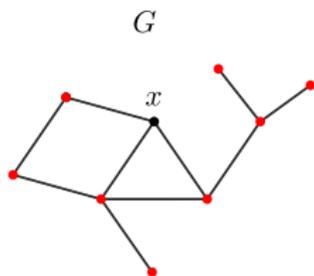
$$\frac{1}{4} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{pmatrix}$$

Definition (Kron reduction)

Let $G = (V, E, c)$ and $A \subseteq V$ nonempty. The Kron reduction G/A^c is the unique graph on A with Laplacian matrix L/A^c . Equivalently, $G' = G/A^c$ is the unique graph on A such that $\omega'(u, v) = \omega(u, v)$ for all $u, v \in A$.

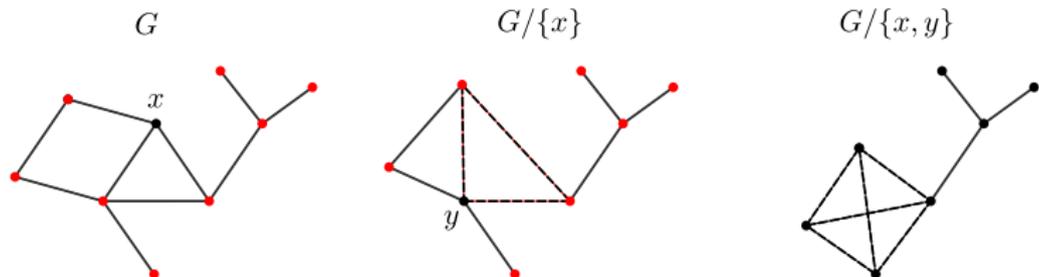
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Note: The Kron reduction $G/\{u, v\}^c$ is an edge (u, v) with weight ω_{uv}^{-1} .

► Application 3: Laplacian submatrix

$$-\frac{1}{2} \begin{pmatrix} 4\sigma^2 & -2p_a^T & -2p_{a^c}^T \\ -2p_a & L_{aa} & L_{aa^c} \\ -2p_{a^c} & L_{a^c a} & L \setminus \{a\} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1^T \\ 1 & 0 & \Omega_{aa^c} \\ 1 & \Omega_{a^c a} & \Omega_{a^c a^c} \end{pmatrix}^{-1}$$

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Proposition (Fiedler '97)

The inverse of a Laplacian submatrix has entries

$$(L \setminus \{a\})_{uv}^{-1} = \frac{1}{2}(\omega_{ua} + \omega_{av} - \omega_{uv})$$

RHS is also called triangle excess or Gromov product

Metric theory of ω

$\omega_{ua} + \omega_{av} \geq \omega_{uv}$ with equality iff a is a cut vertex between u, v (\star)

Corollary

The inverse Laplacian submatrix $(L \setminus \{a\})^{-1}$ is block-diagonal with positive blocks corresponding to the connected components of $G \setminus \{a\}$.

Metric theory of ω

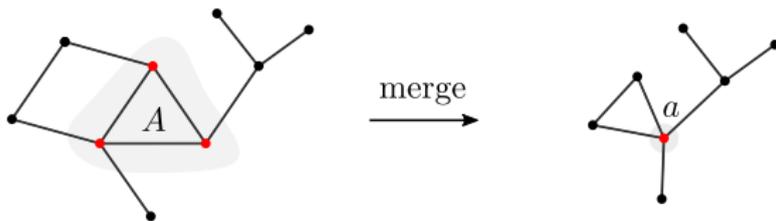
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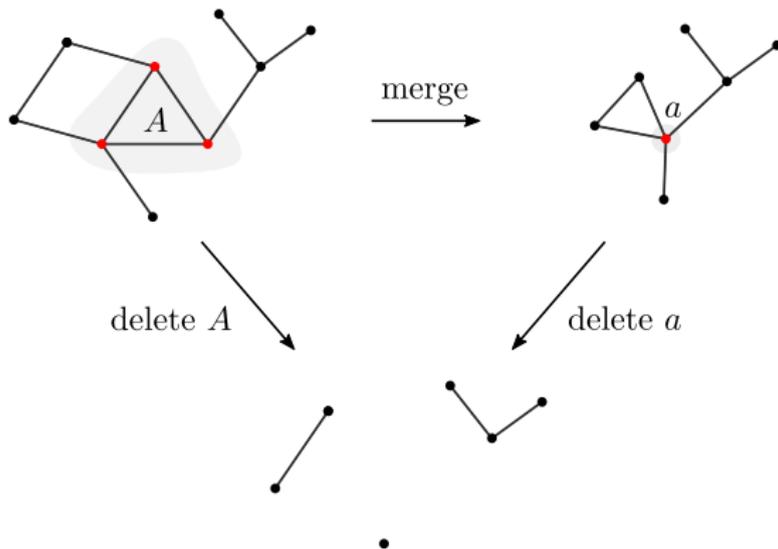
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Note: Can prove this corollary with matrix theory ($L \setminus \{a\}$ is symmetric non-singular M -matrix), and then prove (\star).

Application 3+1: vertex-merged graphs



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$$\text{delete} = \text{delete} \circ \text{merge}$$

$$L \setminus A = L' \setminus \{a\}$$

$$-\frac{1}{2} \begin{pmatrix} 4\sigma^2 & -2p_A^T & -2p_{Ac}^T \\ -2p_A & L_{AA} & L_{AAc} \\ -2p_{Ac} & L_{AcA} & L_{AcAc} \end{pmatrix} = \begin{pmatrix} 0 & 1^T & 1^T \\ 1 & \Omega_{AA} & \Omega_{AAc} \\ 1 & \Omega_{AcA} & \Omega_{AcAc} \end{pmatrix}^{-1}$$

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Proposition (D. 2022)

Let G be a graph, G' the $A \rightarrow a$ merged graph and $G'' = G/A^c$ the Kron reduction. Then **the effective resistances in G' are equal to**

$$\omega'_{uv} = \omega_{uv} - \frac{1}{4} \sum_{i \sim'' j} c''_{ij} (\omega_{iu} + \omega_{jv} - \omega_{iv} - \omega_{ju})^2$$

$$\omega'_{av} = -\sigma''^2 + \sum_{u \in A} \omega_{au} p''_a - \frac{1}{4} \sum_{u \sim'' w} c''_{uw} (\omega_{au} - \omega_{uw})^2$$

► Application 3+2: Characterization of resistance matrices

Theorem (D. 2022)

A real symmetric invertible matrix Ω is a resistance matrix if and only if

$$(1^T \Omega^{-1} 1)(\Omega^{-1})_{ij} \geq (\Omega^{-1} 1)_i (\Omega^{-1} 1)_j \text{ for all } i \neq j$$

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Proof idea: Following Bapat's theorem, define

$$\tilde{L} = \Omega^{-1} - \frac{\Omega^{-1} 1 1^T \Omega^{-1}}{1^T \Omega^{-1} 1}$$

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Definition

A resistance matrix Ω is a real symmetric invertible matrix such that

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► Another characterization:

relative resistance $r_e := c_{uv}\omega_{uv}$ for $e = (u, v) \in E(G)$

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Let $G = (V, E)$ be a biconnected graph. Then $(r_e)_{e \in E}$ are the relative resistances in G with respect to some weighting $c > 0$ if and only if

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- $\sum_{e \in A} r_e < n - \# \text{components of } G[A]$ for all $A \subset E$.

Furthermore, the weights are unique up to common scaling.

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Conditions (1)–(3) determine the **spanning tree polytope**

$$P(G) := \text{conv} (e_T : T \subseteq E \text{ is a spanning tree of } G) \subset \mathbb{R}^{|E|}$$

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$$P(G) := \text{conv} (e_T : T \subseteq E \text{ is a spanning tree of } G) \subset \mathbb{R}^{|E|}$$

All possible relative resistances of G form the interior of $P(G)$

$$\begin{array}{ccccc}
 L_{AA} & \longleftarrow & L & \longrightarrow & L/A^c \\
 & \updownarrow & & \updownarrow & \\
 & & \Omega & \longrightarrow & \Omega_{AA} \\
 \Omega/A^c & \longleftarrow & & &
 \end{array}$$

- ▷ Bapat's identity for Ω^{-1}
- ▷ Kron reduction, $G/A^c \leftrightarrow L/A^c \leftrightarrow \Omega_{AA}$
- ▷ Laplacian submatrix inverse
- ▷ ω' in vertex-merged graphs
- ▷ Characterization of resistance matrices

Invariants p and σ^2

Fiedler–Bapat identity introduces two invariants:

$$\sigma^2 := \frac{1}{2} (1^T \Omega^{-1} 1)^{-1} \quad \text{and} \quad p = \frac{\Omega^{-1} 1}{1^T \Omega^{-1} 1} \quad \Rightarrow \quad \Omega p = 2\sigma^2 1$$

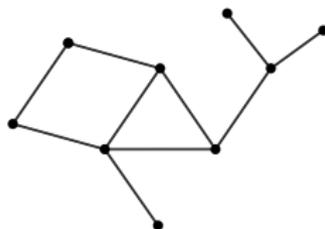
Invariants p and σ^2

Fiedler–Bapat identity introduces two invariants:

$$\sigma^2 := \frac{1}{2} (1^T \Omega^{-1} 1)^{-1} \quad \text{and} \quad p = \frac{\Omega^{-1} 1}{1^T \Omega^{-1} 1} \Rightarrow \Omega p = 2\sigma^2 1$$

Equivalently

$$\sigma^2 = \frac{1}{4} \sum_{u \sim v} c_{uv} (\omega_{u\mathbf{x}} - \omega_{v\mathbf{x}})^2 \quad \text{and} \quad p_v = 1 - \frac{1}{2} \sum_{u \sim v} c_{uv} \omega_{uv}$$



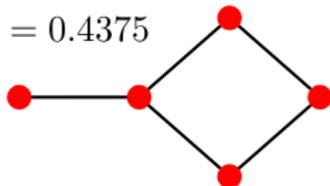
► Some results on σ^2

$\sigma^2(A) :=$ take Kron reduction G/A^c and calculate σ^2

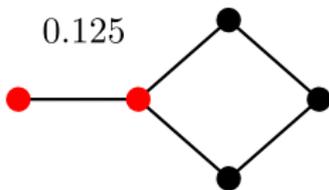
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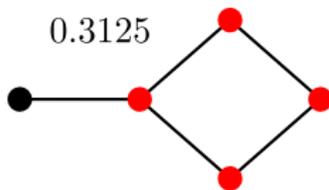
$$\sigma^2(\bullet) = 0.4375$$



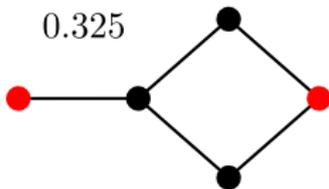
$$0.125$$



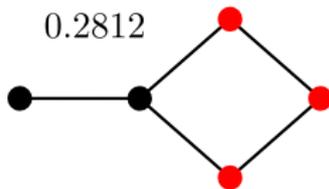
$$0.3125$$



$$0.325$$



$$0.2812$$



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Proposition (D. 2022)

- if $A \subseteq B$ then $\sigma^2(A) \leq \sigma^2(B)$

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- if $A \cap B$ is a cut set, then $\sigma^2(A \cup B) = \sigma^2(A) + \sigma^2(B) - \sigma^2(A \cap B)$

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- if $p \geq 0$ then $\sigma^2(A \cup B) \leq \sigma^2(A) + \sigma^2(B) - \sigma^2(A \cap B)$ for all $A \cap B \neq \emptyset$ ($=$: submodular^{*})

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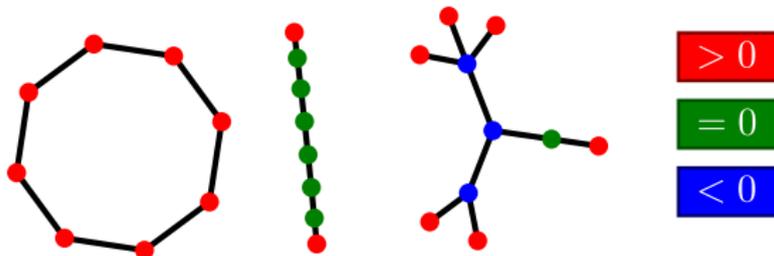
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- if σ^2 is submodular * , then $p \geq -\frac{1}{2}$

Open problem: Do we have “submodular * $\Rightarrow p \geq 0$ ”

► Some results on p



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Let G be connected with $p > 0$. Then

- $p' > 0$ in every Kron reduction G' ,

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- G or $G \setminus \{v\}$ for any v has a perfect matching.

Open problem: Study the class of unweighted graphs which admit a weighting c such that $p \geq 0$.

Conjecture [Fiedler 2011]: These are precisely 1-tough graphs

Research question: L is a singular, symmetric M -matrix. Which parts of the theory can we generalize?

- 1 Do we get a Fiedler–Bapat identity?
- 2 What do Schur complements and submatrices mean?
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Thank you! Questions?

karel.devriendt@mis.mpg.de

- ▷ Devriendt “*Effective resistance is more than distance: Laplacians, Simplices and the Schur complement*”, 2022, LAA
- ▷ Devriendt, Lambiotte “*Discrete curvature on graphs from the effective resistance*”, 2022, Journal of Physics: Complexity
- ▶ Devriendt “*Graph Geometry from effective resistances*”, 2022, PhD thesis - available online

Appendix 1: Exercises

Exercise 1: Calculate Ω for small example by PC (*hint: see next slide*)

Exercise 2: What is the Kron reduction G/A^c if $A = \{u, v\}$? (*hint: what happens to Ω ?*)

Exercise 3: Give a matrix-theoretic proof of this corollary and use it to prove statement (\star), ie triangle inequality for ω
(*hint: $L \setminus \{a\}$ is symmetric non-singular M -matrix*)

Exercise 4: Calculate ω' when merging two vertices $\{a, b\} \rightarrow c$.
(*hint: G'' is a single edge with $\sigma''^2 = \omega_{ab}/4$ and $p = 1/2$)*

Exercise 5: Prove equivalence between different expressions of p and σ^2
(*hint: $Q\Omega = -2I + 2p1^T$)* and calculate p and σ^2 for the complete graph $K_4 = \square$

E-mail me for solutions: karel.devriendt@mis.mpg.de

Appendix 2: Computing Ω in practice

In practice, we compute Ω via the Moore-Penrose pseudoinverse Laplacian (see eg Wikipedia). The pseudoinverse Laplacian is denoted by L^\dagger .

The effective resistance is then computed by:

$$\omega_{uv} = (L^\dagger)_{uu} + (L^\dagger)_{vv} - 2(L^\dagger)_{uv}$$

The resistance matrix is computed by:

$$\Omega = \mathbf{1}\zeta^T + \zeta\mathbf{1}^T - 2L^\dagger, \text{ where } \zeta = \text{diag}(L^\dagger)$$

Most linear algebra software has a function *pinv* for pseudoinverse. Matlab code for small example:

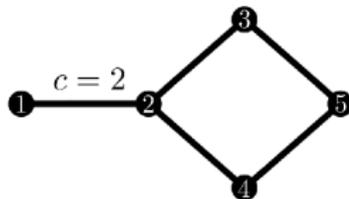
```
>> %Input the Laplacian matrix (L)
L = [2, -2, 0, 0, 0; -2, 4, -1, -1, 0; 0, -1, 2, 0, -1; 0, -1, 0, 2, -1; 0, 0, -1, -1, 2]
%Calculate the pseudoinverse (pL)
pL = pinv(L);
%Get the diagonal of the pseudoinverse (zeta) and a ones-vector (u)
zeta = diag(pL);
u = ones(size(zeta));
%Calculate the resistance matrix (Omega)
Omega = u*zeta' + zeta*u' - 2*pL
```

L =

2	-2	0	0	0
-2	4	-1	-1	0
0	-1	2	0	-1
0	-1	0	2	-1
0	0	-1	-1	2

Omega =

0	0.5000	1.2500	1.2500	1.5000
0.5000	0	0.7500	0.7500	1.0000
1.2500	0.7500	0	1.0000	0.7500
1.2500	0.7500	1.0000	0	0.7500
1.5000	1.0000	0.7500	0.7500	0



```
>> %calculate p and sigma^2
sigma = 1/2*inv(u'*inv(Omega)*u)
p = inv(Omega)*u/(u'*inv(Omega)*u)

sigma =

    0.4375

p =

    0.5000
   -0.2500
    0.2500
    0.2500
    0.2500
```