An overview of the Calderón problem

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INTRODUCTION

Goal: to determine the electrical conductivity of a medium from voltage and current measurements on the boundary.

Applications. oil prospection, non-destructive testing (ERT)
medical imaging (EIT)

Formulation: $\Omega \subset \mathbb{R}^n$ bdd, regular enough.
$\gamma \in L^\infty(\Omega)$: $\gamma(x) \geq c > 0$

Conductivity eq.

$$\left.\begin{aligned} -\nabla\cdot(\gamma\nabla v) &= 0 \quad \text{in } \Omega \\ v &= g \quad \text{on } \partial\Omega \end{aligned}\right\} \quad (1)$$

Dirichlet-to-Neumann map:

$$\Lambda_\gamma : g \longmapsto \underbrace{\gamma \partial_\nu v|_{\partial\Omega}}_{\text{current flux through } \partial\Omega} \quad (\text{weak sense})^*$$

Well-possed $\longrightarrow$ Well-defined & bdd.

$g \in H^{1/2}(\partial\Omega) \Rightarrow \exists!\, v \in H^1(\Omega)$ $\qquad$ $H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$

$* \langle \Lambda_\gamma g, h\rangle = \int_\Omega \gamma \nabla v \cdot \nabla e_h \, dx \quad \forall h \in H^{1/2}(\partial\Omega)$

Calderón problem: $\Lambda_\gamma \longmapsto \gamma$

Main questions:

- Uniqueness: $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$?
- Reconstruction: $\exists$ convergent algorithm $\Lambda_\gamma \rightsquigarrow \gamma$?
- Stability. $\Lambda_{\gamma_1}, \Lambda_{\gamma_2}$ close $\Rightarrow \gamma_1, \gamma_2$ close ?

Some features.

- Non-linear.
- Ill-posed. No stability; only if $\gamma_i \in K$ and even so, no Lip. dependence.
- $n=1$ undetermined; $n=2$ determined; $\underline{\underline{n \geq 3}}$ overdetermined

Further questions:

- Finite / discrete measurements.
- Partial data
- Stabilization effects.
- ⊕ Anisotropic conductivities.

Bibliography (short surveys)

- M. Salo, Calderón problem (Lecture notes)

  http://users.jyu.fi/~salomi/lecturenotes/calderon_lectures.pdf

- G. Uhlmann, 30 years of the Calderón's problem

  http://www.numdam.org/item/10.5802/slsedp.40.pdf

THE CALDERÓN PROBLEM FOR THE SCHRÖDINGER EQ.

Formulation: $q \in L^\infty(\Omega)$

$$\left.\begin{aligned} (-\Delta + q)u &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega \end{aligned}\right\} (2) \qquad \Lambda_q : f \longmapsto \partial_\nu u|_{\partial\Omega}$$

Well-posedness needs to be assume (0 not an eigenvalue).

Inverse problem: $\Lambda_q \longrightarrow q$

Reduction of the EIT problem if $\gamma \in C^2(\bar{\Omega})$

(1) and (2) are equivalent with $q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \in L^\infty(\Omega)$, $f = \gamma^{1/2} g$, $u = \gamma^{1/2} v$.

Well-posedness for (2) is ensured in this case.

$$\Lambda_q f = \gamma^{1/2} \Lambda_\gamma (\gamma^{-1/2} f) + \frac{1}{2} \gamma^{-1} (\partial_\nu \gamma) f \Big|_{\partial\Omega}$$

$$\Lambda_\gamma \xrightarrow[\text{Boundary reconstruction [Kohn, Vogelius 1984]}]{} \Lambda_q \longrightarrow q \xrightarrow[\substack{(-\Delta+q)\gamma^{1/2}=0 \text{ in } \Omega \\ \text{Boundary reconstr.} \to \gamma^{1/2}|_{\partial\Omega}}]{} \gamma$$

Main ideas:

- **Alessandrini's identity:**

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx, \qquad \left.\begin{aligned} (-\Delta + q_i) u_i &= 0 \quad \text{in } \Omega \\ u_i &= f_i \quad \text{on } \partial\Omega \end{aligned}\right\}$$

- **Uniqueness** (Sylvester, Uhlmann 1987)

$$\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow \int_{\Omega \to \mathbb{R}^n} (q_1 - q_2) u_1 u_2 \, dx = 0 \Rightarrow q_1 = q_2$$

$$u_1 u_2 \sim e^{i\xi \cdot x} \quad \forall \xi \in \mathbb{R}^n$$

- **Complex geometric optic solutions** (CGOs)

$u = e^{i\zeta \cdot x}(1 + r_\zeta)$ with $\zeta \in \mathbb{C}^n$: $\zeta \cdot \zeta = 0$ & "$r_\zeta \to 0$" as $|\zeta| \to \infty$.

$\Rightarrow \zeta_1, \zeta_2 \in \mathbb{C}^n$: $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$, $|\zeta_1|, |\zeta_2| \to \infty$, $\zeta_1 + \zeta_2 = \xi$ $\Rightarrow n \geq 3$

Then $u_1 u_2 = e^{i\xi \cdot x}(1 + r_{\zeta_1} + r_{\zeta_2} + r_{\zeta_1} r_{\zeta_2}) \longrightarrow e^{i\xi \cdot x}$

CGO solutions

<u>Thm</u> (Sylvester-Uhlmann 1987): $\exists\, C_0 = C_0(n,\Omega)$ s.t. for any $\zeta \in \mathbb{C}^n$: $\zeta\cdot\zeta = 0$, $|\zeta| \geq \max(C_0\|q\|_{L^\infty(\Omega)}, 1)$ $\exists\, u \in H^2(\Omega)$ s.t.

- $u(x) = e^{i\zeta\cdot x}(1 + r_\zeta(x))$
- $(-\Delta + q)u = 0$ in Ω
- $\|r_\zeta\|_{H^m(\Omega)} \leq C C_0 |\zeta|^{m-1} \|q\|_{L^\infty(\Omega)}$ $\quad m = 0,1,2.$

<u>Idea of the proof</u>:

$$e^{-i\zeta\cdot x}(-\Delta + q)(e^{i\zeta\cdot x}(1+r_\zeta)) = (-\Delta - 2i\zeta\cdot\nabla + q)(1+r_\zeta) = 0$$

$$\Rightarrow \boxed{(-\Delta - 2i\zeta\cdot\nabla + q)\, r_\zeta = -q}$$

Step 1: Constant coefficients: $(-\Delta - 2i\zeta\cdot\nabla)\, r = \underset{\substack{\cup\\ L^2(\Omega)}}{F}$ in $\underset{\substack{\cap\\ \Omega}}{(-\pi,\pi)^n}$

Fourier with a shift $\quad e^{ik\cdot x} \underset{\zeta = \tau(e_1 + ie_2)}{\rightsquigarrow} e^{i(k + \frac{1}{2}e_2)\cdot x} \quad (k \in \mathbb{Z}^n)$

$$|k|^2 + 2\zeta\cdot k \qquad |k + \tfrac{1}{2}e_2|^2 + 2\tau\left(k_1 + i(k_2 + \tfrac{1}{2})\right) \neq 0$$

$\|r\|_{H^m(\Omega)} \leq C_0 |\zeta|^{m-1} \|F\|_{L^2(\Omega)} \qquad |\zeta| > C \qquad$ (constrain in H^1, H^2 norms)

Step 2: $(-\Delta - 2i\zeta\cdot\nabla + q)\, r = F$

$$(-\Delta - 2i\zeta\cdot\nabla)\, r = F - qr = \tilde{F} \quad \Rightarrow \quad r = G_\zeta \tilde{F}$$

$$\tilde{F} = \underbrace{(I + qG_\zeta)^{-1}}_{\text{invertible if } \|qG_\zeta\|_{L^2\to L^2} < 1} F$$

$$\|q\|_{L^\infty} C_0 |\zeta|^{-1} \leq \tfrac{1}{2}$$

A few (more) ideas about the main results $\quad (\underline{n \geq 3})$

$q_j \in L^\infty(\Omega)$: 0 not an eigenvalue of $-\Delta + q_j$ in Ω.

Uniqueness (Sylvester, Uhlmann 1987)

$$\boxed{\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2}$$

Given $\xi \in \mathbb{R}^n$, let $\theta_1, \theta_2 \in \mathbb{R}^n$: $\theta_1 \cdot \theta_2 = \theta_1 \cdot \xi = \theta_2 \cdot \xi = 0$

$$|\theta_2|^2 = |\theta_1|^2 + |\xi|^2$$

$$\left.\begin{array}{l} \zeta_1 = \frac{1}{2}(\xi + \theta_1 + i\theta_2) \\ \zeta_2 = \frac{1}{2}(\xi - \theta_1 - i\theta_2) \end{array}\right\} \quad u_1 u_2 \longrightarrow e^{i\xi\cdot x} \quad \text{as } |\theta_1| \longrightarrow \infty$$

Stability (Alessandrini 1988)

$$\boxed{\|q_j\|_{L^\infty(\Omega)} \leq M} \Rightarrow \boxed{\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C\,|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_*}$$

$$\int_{\mathbb{R}^n} (q_1 - q_2) e^{ix\cdot\xi}\, dx = \langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle + \int_{\mathbb{R}^n} (q_1 - q_2) e^{ix\cdot\xi} (r_{\zeta_1} + r_{\zeta_2} + r_{\zeta_1} r_{\zeta_2})\, dx$$

$$|\widehat{q_1 - q_2}(\xi)| \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_* \underbrace{\|u_1\|_{H^1(\Omega)}}_{\leq c|\zeta_1| e^{c|\zeta_1|}} \underbrace{\|u_2\|_{H^1(\Omega)}}_{\leq c|\zeta_2| e^{c|\zeta_2|}} + M\,(\underbrace{\|r_{\zeta_1}\|_{L^2(\Omega)} + \ldots}_{\leq c(|\zeta_1|^{-1} + |\zeta_2|^{-1} + |\zeta_1|^{-1}|\zeta_2|^{-1})})$$

$|\zeta_j| = \tau$

$$\lesssim \|\Lambda_{q_1} - \Lambda_{q_2}\| \tau e^{c\tau} + \tau^{-1} \qquad \text{Optimization } \epsilon e^{c\tau} \simeq \tau^{-1}$$

* Optimal (Mandache 2001)

Reconstruction (Nachmann 1988) $\quad \Omega$ smooth

$$\langle (\Lambda_q - \Lambda_0)\underbrace{(u_{\zeta_1}|_{\partial\Omega})}_{e^{i\zeta_1\cdot x}(1 + \underbrace{r_{\zeta_1}}_{?})|_{\partial\Omega}},\ e^{i\zeta_2\cdot x}|_{\partial\Omega} \rangle = \int q\, u_{\zeta_1} e^{i\zeta_2\cdot x} \xrightarrow[|\theta_1|\to\infty]{} \int q e^{i\xi\cdot x}$$

$$\underbrace{(-\Delta - 2i\zeta\cdot\nabla)}_{e^{-i\zeta\cdot x}(-\Delta)e^{i\zeta\cdot x}} r = -q(1 + r) \Rightarrow (-\Delta) e^{i\zeta\cdot x} r = -q u_\zeta$$

$$\Rightarrow e^{i\zeta\cdot x} r = -\int_\Omega \underbrace{K_\zeta(x,y)}_{-\Delta K_\zeta = \delta_x} q u_\zeta = \langle (\Lambda_q - \Lambda_0) u_\zeta|_{\partial\Omega}, K_\zeta|_{\partial\Omega} \rangle$$

$x \notin \Omega \quad \rightsquigarrow \quad x \to \partial\Omega.$

Some further results.

- Uniqueness for Lip. conductivities [Caro, Rogers 2016]

  q as a distribution, suitable a priori estimates

- Lip. stability for piecewise constant conductivities [Alessandrini, Vessella 2005]

- Finite number of measurements for $q(\gamma)$ in a finite dimensional subspace. (Lip. conductivity)

  [Alberti, Santacesaria 2018]

- Born approximation (linear on Λ_q)

$$\hat{q}_B(\xi) = \lim_{|\xi_1| \to \infty} \langle (\Lambda_q - \Lambda_0) e^{i\xi_1 \cdot x}\big|_{\partial\Omega}, e^{i\xi_2 \cdot x}\big|_{\partial\Omega} \rangle$$

  Explicit formula in terms of Λ_q-matrix for $\Omega = B^3$

  [Barceló, Castro, Macià, Meroño 2022]

THE CALDERÓN PROBLEM FOR NONLOCAL SCHRÖDINGER EQS.

Fractional Laplacian

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\mathcal{F}u)$$

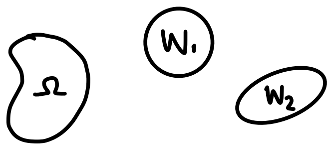
$$(-\Delta)^s u(x) = \text{p.v.} \int_{\mathbb{R}^n} (u(x+y) - u(x))\, |y|^{-n-2s}\, dy$$

Random walk with possibly long jumps; Levy process

$$\left.\begin{aligned} ((-\Delta)^s + q)\, u &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{in } \Omega_e \end{aligned}\right\} (3) \qquad \Lambda_q : f \longmapsto (-\Delta)^s u|_{\Omega_e}$$

Uniqueness from partial data: [Ghosh, Salo, Uhlmann 2020]

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_1} f|_{W_2} \quad \forall f \in C_0^\infty(W_1) \Rightarrow q_1 = q_2$$



- $\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle = \int_\Omega (q_1 - q_2)\, u_1 u_2$
- Runge approximation: $\{u|_\Omega : (3) \text{ with } f \in C_0^\infty(W)\}$ dense in $L^2(\Omega)$.
- Antilocality: $u = 0 = (-\Delta)^s u$ in $\Omega \Rightarrow u \equiv 0$. (AL)

Uniqueness from one measurement [G, Rüland, S, U 2020] $q_i \in C^0(\bar{\Omega})$

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_1} f|_{W_2} \text{ for some } f \in C_0^\infty(W_1)\setminus\{0\} \Rightarrow q_1 = q_2$$

$$u_1 - u_2 = (-\Delta)^s (u_1 - u_2) = 0 \text{ in } W_2 \overset{AL}{\Rightarrow} u_1 - u_2 = 0 \text{ in } \Omega$$

$$q_1 \overset{*}{=} \frac{(-\Delta)^s u_1}{u_1} = \frac{(-\Delta)^s u_2}{u_2} \overset{*}{=} q_2$$

* Fine if $u_i \neq 0$ in $U \subset \Omega$.
True due to antilocality.

Reconstruction

$$u = \lim_{t \to 0} u_t \qquad u_t := \arg\min_{w \in \tilde{H}^s(\Omega)} \left\{ \| (-\Delta)^s w - \Lambda_q f \|^2_{H^{-s}(W)} + t \|w\|^2_{H^s(\mathbb{R}^n)} \right\}$$

(Tikhonov regularization)