M-matrices and Randow Walks

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IV Jornadas ALAMA: Laplacian and M-matrices in graphs

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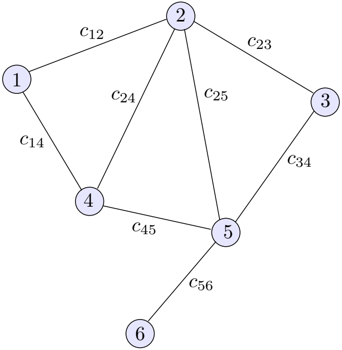
BARCELONATECH

Take me for a (simple) random walk

- Let $\Gamma = (V, E, c)$ be a connected network, $|V| = n$ (n=6) and $|E| = m$ (m=8).



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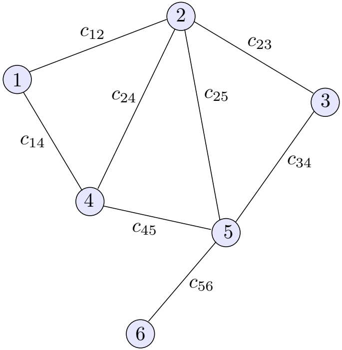
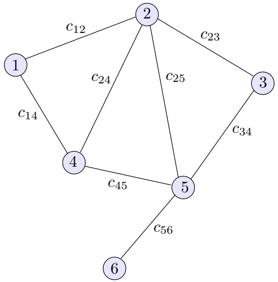


- Given an initial vertex s_0, select *at random* an adjacent vertex s_1, and move to this neighbour, and after to s_2 neighbour of s_1 and so on.

Take me for a (simple) random walk

- The sequence of vertices $s_0, s_1, s_2, \ldots, s_k, \ldots$ is a simple RW on Γ.



- At each step k, we have a random variable X_k taking values on V.

- Discrete time stochastic process defined on the state space V

$$X_0, X_1, X_2, \ldots, X_k, \ldots$$

Random Walks & Markov chains: Just the same

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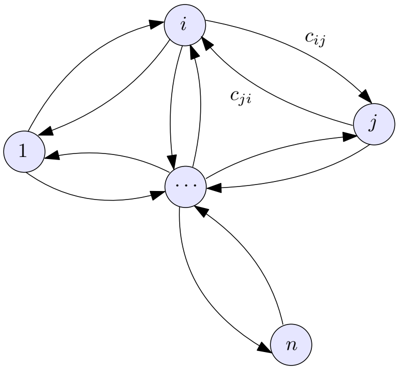
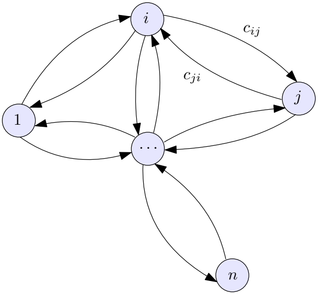
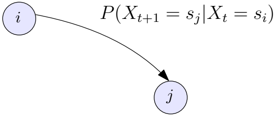


Random Walks & Markov chains: Just the same

◦ Let $\Gamma = (V, E, c)$ be a connected network, $|V| = n$ and $|E| = m$.

▶ What does *at random* mean?





If at time k we are at vertex i, choose any neighbour to move to.

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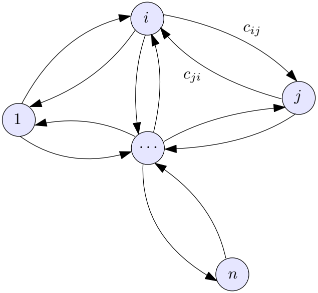

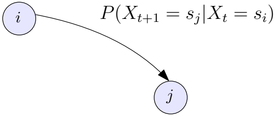

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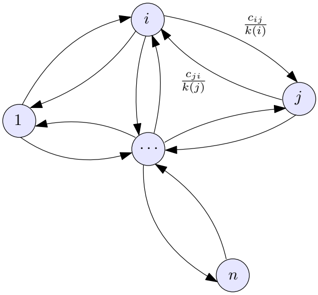
- Markov: $P\big(X_{k+1} = j | X_k = i, X_{k-1} = i_{k-1}, \dots, X_0 = i_0\big) = P\big(X_{k+1} = j | X_k = i\big)$

Take chances on the RW

- Let $k_i = \sum_{j \sim i} c_{ij}$ denote the degree of vertex i then



$$p_{ij} = P\big(X_{k+1} = j | X_k = i\big) = \frac{c_{ij}}{k_i}$$

- Transition probability matrix: the stochastic matrix $\mathsf{P} = \big(p_{ij}\big)_{i,j=1}^{n}$

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- The ij-th entry of P^k represents the probability of reaching state j after k steps if the walk stars at state i:

$$\mathsf{u}_k^\mathsf{T} = \mathsf{u}_0^\mathsf{T}\mathsf{P}^k$$

$\mathsf{u}_t^\mathsf{T} = \big(P(X_t = 1), P(X_t = 2), \ldots, P(X_t = n)\big)$ is the probability distribution vector at step t.

Take chances on the RW

- Transition probability matrix: the stochastic matrix $\mathsf{P} = \big(p_{ij}\big)_{i,j=1}^{n}$

- The Perron-Frobenius theorem for nonnegative

  irreducible matrices $\equiv$ connected network

  implies the existence of a unique probability distribution $\boldsymbol{\pi}$, which is a positive left eigenvector associated to its dominant eigenvalue $\lambda = 1$.

$$\boldsymbol{\pi}^{\mathsf{T}} = \boldsymbol{\pi}^{\mathsf{T}}\mathsf{P}, \qquad \pi_i > 0, \qquad \sum_{i \in V} \pi_i = 1$$

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- Stationary distribution, π:

$P(X_0 = j) = \pi_j$, $j = 1, \dots, n$, then $P(X_k = j) = \pi_j$, for all k.

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- Stationary distribution, π:

$$\pi_i = \frac{k_i}{\mathrm{vol}(\Gamma)}, \text{ for all } i \in V, \text{ where } \mathrm{vol}(\Gamma) = \sum_{i,j\in V} c_{ij} = \sum_{i\in V} k_i.$$

Let's talk about the long term behavior of the RW

- Consider $\Pi = 1 \otimes \boldsymbol{\pi} = 1 \cdot \boldsymbol{\pi}^\top = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & \vdots & \dots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$, then

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Regular random walks

$$\Pi = \lim_{n \to +\infty} \mathsf{P}^n$$

Some power n of P does not have zeros

$\Downarrow$

It is possible for all i to go to all j in exactly n steps

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Regular random walks

$$\Pi = \lim_{n \to +\infty} \mathsf{P}^n$$

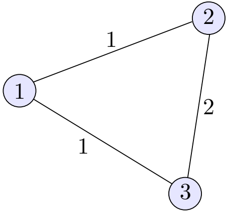
but FOR JUST

Irreducible random walks

$$\Pi = \lim_{\mathsf{n} \to +\infty} \frac{\mathsf{I} + \mathsf{P} + \ldots + \mathsf{P}^{\mathsf{n}-1}}{\mathsf{n}}$$

Let's talk about the long term behavior of the RW

Example regular randow walk:



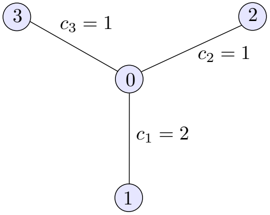
$$\mathsf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \quad \mathsf{P}^{20} = \begin{bmatrix} 0.25 & 0.375 & 0.375 \\ 0.25 & 0.375 & 0.375 \\ 0.25 & 0.375 & 0.375 \end{bmatrix}$$

$$\lim_{n\to\infty} \mathsf{P}^n = \Pi = \mathsf{1} \otimes \boldsymbol{\pi} = \mathsf{1} \cdot \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$$

$$= \mathsf{1} \otimes \frac{1}{\sum\limits_{i\in V} k_i} \mathsf{k} = \mathsf{1} \cdot \frac{1}{8} \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$$

Let’s talk about the long term behavior of the RW

Example JUST irreducible randow walk, not regular:

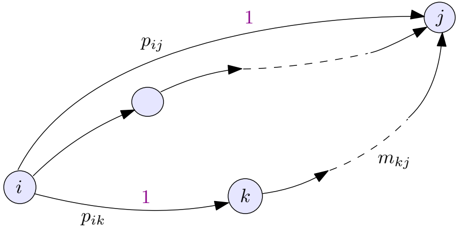


$$\mathsf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi = \lim_{n\to+\infty} \frac{\mathsf{I} + \mathsf{P} + \ldots + \mathsf{P}^{n-1}}{n} \quad \text{and} \quad \boldsymbol{\pi} = \frac{1}{\sum\limits_{i\in V} k_i}\,\mathsf{k} = \frac{1}{8}\begin{bmatrix} 4 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

What about short term behavior?

- If a random walk starts in state s_i, the expected number of steps to reach state s_j for the first time is called the Mean First Passage Time or the hitting time from s_i to s_j. It is denoted by m_{ij}.



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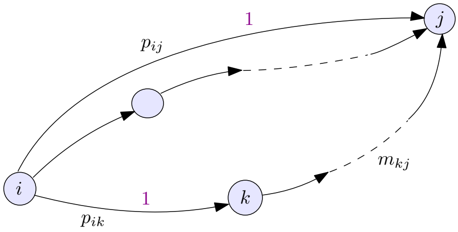

$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}(m_{kj} + 1) = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

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$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}(m_{kj} + 1) = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

◦ If $i = j$, the expected number of steps to return to s_i for the first time is the mean recurrence time for s_i.

$$m_{ii} = \frac{1}{\pi_i}$$

Randow Walks meet M-matrices

◦ Probabilistic Laplacian and combinatorial Laplacian

$$\Delta = \mathsf{I} - \mathsf{P} \qquad \mathsf{L} = \mathsf{D_k} - \mathsf{A} = \mathsf{D_k}\,(\mathsf{I} - \mathsf{P})$$

where $\mathsf{D_k}$ is diagonal of degrees and $\mathsf{A} = (c_{ij})_{i,j=1}^{n}$ is the adjacency matrix.

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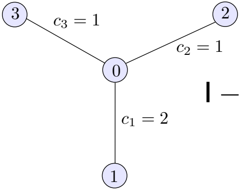

$$\mathsf{I} - \mathsf{P} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{L} = \begin{bmatrix} 4 & -2 & -1 & -1 \\ -2 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

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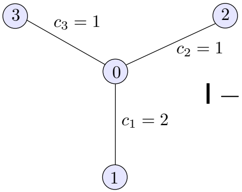

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- Properties

$$\Delta 1 = (\mathsf{I} - \mathsf{P})\, 1 = 0 \quad \text{and} \quad \mathsf{L}1 = 0$$

L is a symmetric positive (semi)-definite Z-matrix $\Rightarrow$ L is an M-matrix

Back to Mean First Passage Time

$$m_{ij} = p_{ij} + \sum_{k\neq j} p_{ik}(m_{kj}+1) = 1 + \sum_{k\neq j} p_{ik} m_{kj} \quad \text{and} \quad m_{ii} = \frac{1}{\pi_i}$$

- The mean first passage time matrix, $\mathsf{M} = (m_{ij})$:

$$\mathsf{M} = \mathsf{P}\mathsf{M} + \mathsf{J} - \mathsf{P}\mathsf{D}_\pi^{-1} \iff \big(\mathsf{I} - \mathsf{P}\big)\mathsf{M} = \mathsf{J} - \mathsf{P}\mathsf{D}_\pi^{-1}$$

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◦ The mean first passage time matrix, $\mathsf{M} = (m_{ij})$:

$$\mathsf{M} = \mathsf{PM} + \mathsf{J} - \mathsf{P}\mathsf{D}_\pi^{-1} \iff \big(\mathsf{I} - \mathsf{P}\big)\mathsf{M} = \mathsf{J} - \mathsf{P}\mathsf{D}_\pi^{-1}$$

◦ Properties of the system

- $\mathsf{I} - \mathsf{P}$ is singular
- The system is compatible
- $(\mathsf{I} - \mathsf{P}) \cdot \mathsf{1} = 0 \Rightarrow \mathsf{1} \in \mathrm{Ker}(\mathsf{I} - \mathsf{P})$

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◦ Any 1-inverse or *system solving* inverse, H, of Δ provide a solution.

$$\mathsf{M} = \mathsf{H}\big(\mathsf{J} - \mathsf{P}\mathsf{D}_\pi^{-1}\big) + \mathsf{1}\alpha^{\mathsf{T}}$$

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- Imposing the condition $m_{ii} = \frac{1}{\pi_i}$ ($\mathsf{M}_\mathsf{d} = \mathsf{D}_\pi^{-1}$):

$$\mathsf{M} = \big(\mathsf{H}\Pi - \mathsf{J}(\mathsf{H}\Pi)_d + \mathsf{I} - \mathsf{H} + \mathsf{J}\mathsf{H}_d\big)\mathsf{D}_\pi^{-1}$$

The matrices that we use to know

$$\mathsf{D_k}\big(\mathsf{I}-\mathsf{P}\big)\mathsf{M}=\mathsf{D_k}\big(\mathsf{J}-\mathsf{P}\mathsf{D}_\pi^{-1}\big)\iff\mathsf{L}\mathsf{M}=\mathsf{D_k}\mathsf{J}-\mathrm{vol}(\Gamma)\mathsf{A}\mathsf{D}_\mathsf{k}^{-1}$$

- Consider G any 1-inverse of L

$$\mathsf{M}=\mathsf{G}\mathsf{D_k}\mathsf{J}-\mathsf{J}(\mathsf{G}\mathsf{D_k}\mathsf{J})_d+\mathrm{vol}(\Gamma)\big(\mathsf{D_k}^{-1}-\mathsf{G}+\mathsf{J}\mathsf{G}_d\big)$$

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- If G is any 1-inverse of L s.t. $\mathsf{G}\mathsf{k} = g \cdot \mathsf{1}$, g constant

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and for $g = 1$, $\mathsf{G} = \mathsf{Z}_\mathsf{L}$ is invertible and called Fundamental matrix for L

$$\mathsf{Z}_\mathsf{L} = \left(\mathsf{L} + \frac{\mathsf{k}\mathsf{k}^\mathsf{T}}{\mathrm{vol}(\Gamma)}\right)^{-1}.$$

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- If $\mathsf{L}^\#$ is the group inverse for L then,

$$\mathsf{M} = \mathrm{vol}(\Gamma)\Big(\mathsf{D}_\mathsf{k}{}^{-1} - \mathsf{L}^\# + \mathsf{J}\mathsf{L}^\#_d + \Pi\mathsf{L}^\# + \mathsf{L}^\#\Pi^\mathsf{T} - 2\mathsf{J}\mathsf{D}_{\mathsf{L}^\#\pi}\Big)$$

The notorious constant

- The Kemeny's constant is

$$K = \sum_{j \in V} m_{ij}\pi_j \quad \text{or in matrix form} \quad \mathsf{M}\boldsymbol{\pi} = K\mathsf{1}$$

the expected time to get any randomly chosen vertex from vertex i.

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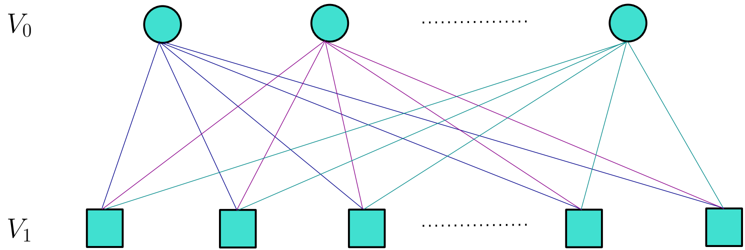
- In terms of $\mathsf{L}^{\#}$, the group inverse for L,

$$K = 1 + \operatorname{tr}\left(\mathsf{L}^{\#}\mathsf{D}_{\mathsf{k}}\right) - \frac{1}{\operatorname{vol}(\Gamma)}\mathsf{k}^{\mathsf{T}}\mathsf{L}^{\#}\mathsf{k}.$$

Let's dance with a complete bipartite graph



$$\mathsf{L}^{\#} = \begin{array}{c} \\ V_0 \\ \\ V_1 \end{array} \left[\begin{array}{c|c} \multicolumn{1}{c}{V_0} & \multicolumn{1}{c}{V_1} \\ \frac{1}{k_0}\left(\mathsf{I} - \frac{n+k_0}{n^2}\mathsf{J}\right) & -\frac{\mathbf{1}}{\boldsymbol{n^2}} \\ \hline -\frac{\mathbf{1}}{\boldsymbol{n^2}} & \frac{1}{k_1}\left(\mathsf{I} - \frac{n+k_1}{n^2}\mathsf{J}\right) \end{array} \right] \qquad n = k_0 + k_1$$

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$$\mathsf{M} = \mathsf{L}^{\#}\mathsf{D}_k\mathsf{J} - \mathsf{J}\left(\mathsf{L}^{\#}\mathsf{D}_k\mathsf{J}\right)_d + \mathrm{vol}(\Gamma)\left(\mathsf{D}_k^{-1} - \mathsf{L}^{\#} + \mathsf{J}\mathsf{L}_d^{\#}\right)$$

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Let's dance with a complete bipartite graph

$$K = 1 + \mathsf{1}^\mathsf{T}\mathsf{L}_d^{\#}\mathsf{k} - \boldsymbol{\pi}^\mathsf{T}\mathsf{L}^{\#}\mathsf{k}$$

Complete bipartite with n vertices then $K = n - \dfrac{1}{2}$

$$K = 1 + \mathsf{1}^\mathsf{T} \left[\begin{array}{c|c} \dfrac{1}{k_0}\left(\mathsf{I} - \dfrac{n+k_0}{n^2}\mathsf{J}\right) & 0 \\ \hline 0 & \dfrac{1}{k_1}\left(\mathsf{I} - \dfrac{n+k_1}{n^2}\mathsf{J}\right) \end{array}\right] \left[\begin{array}{c} k_0 \\ \hline k_1 \end{array}\right]$$

$$- \frac{1}{2k_0k_1} \left[\begin{array}{c|c} k_0 & k_1 \end{array}\right] \left[\begin{array}{c|c} \dfrac{1}{k_0}\left(\mathsf{I} - \dfrac{n+k_0}{n^2}\mathsf{J}\right) & -\dfrac{1}{n^2} \\ \hline -\dfrac{1}{n^2} & \dfrac{1}{k_1}\left(\mathsf{I} - \dfrac{n+k_1}{n^2}\mathsf{J}\right) \end{array}\right] \left[\begin{array}{c} k_0 \\ \hline k_1 \end{array}\right] = n - \frac{1}{2}$$

Let's dance with a complete bipartite graph

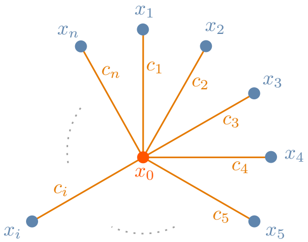
$$K = \sum_{j \in V} m_{ij}\pi_j \quad \text{or in matrix form} \quad K\mathsf{1} = \mathsf{M}\boldsymbol{\pi}$$

$$K\mathsf{1} = \mathsf{M}\boldsymbol{\pi} = \left[\begin{array}{c|c} 2k_1 & 2k_0 - 1 \\ \hline 2k_1 - 1 & 2k_0 \end{array}\right] \frac{1}{2k_0k_1} \left[\begin{array}{c} k_0 \\ \hline k_1 \end{array}\right] = \left(n - \frac{1}{2}\right)\mathbf{1}$$

$$\Downarrow$$

$$K = n - \frac{1}{2}$$

Here comes the network



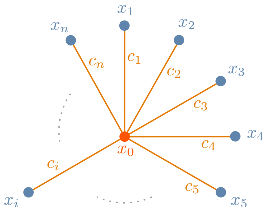
$$\mathsf{L}^{\#}(x_0,x_0) = \frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{c_k} \equiv \alpha$$

$$\mathsf{L}^{\#}(x_0,x_j) = \alpha - \frac{1}{n+1}\frac{1}{c_j}, \qquad j = 1,\dots,n$$

$$\mathsf{L}^{\#}(x_j,x_j) = \alpha - \frac{n-1}{n+1}\frac{1}{c_j}, \qquad j = 1,\dots,n$$

$$\mathsf{L}^{\#}(x_0,x_j) = \alpha - \frac{1}{n+1}\left(\frac{1}{c_i}+\frac{1}{c_j}\right), \qquad i,j = 1,\dots,n,\ i \neq j$$

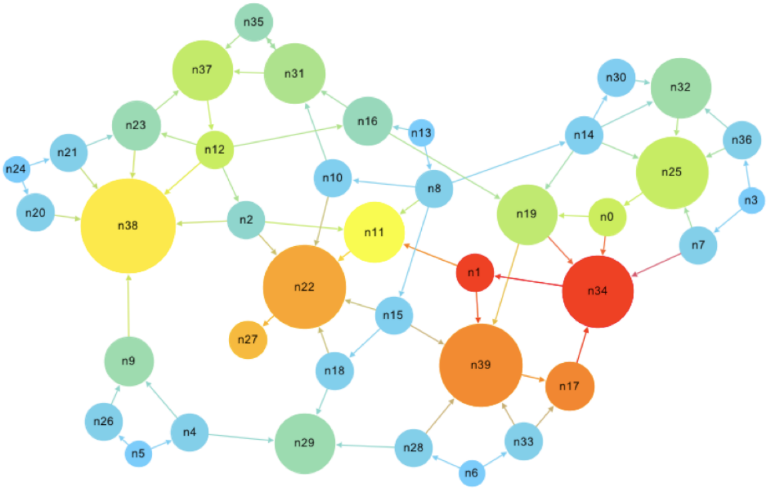
Here comes the network



$$\mathsf{M} = \mathsf{L}^{\#}\mathsf{D}_k\mathsf{J} - \mathsf{J}\big(\mathsf{L}^{\#}\mathsf{D}_k\mathsf{J}\big)_d + \mathrm{vol}(\Gamma)\Big(\mathsf{D}_k^{-1} - \mathsf{L}^{\#} + \mathsf{J}\mathsf{L}_d^{\#}\Big)$$

$$\mathsf{M} = \left[\begin{array}{c|c} \dfrac{1}{\pi_0} & \dfrac{2k_0}{c_j} - 1 \\ \hline 1 & \dfrac{1}{\pi_j} \end{array}\right]$$

Everybody is looking for something

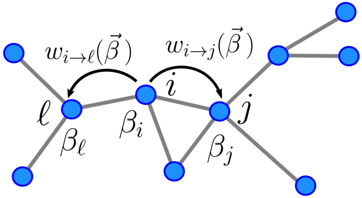


PageRank: Mesure of quantity and quality of websites' links for engine searchers

We do need another models

- Degree-biased random walks describe a dynamics where a walker in a particular network's node uses the degree information of neighbors to choose randomly a new node to visit

$$p_{ij}^{\beta} = \frac{c_{ij} k_j^{\beta}}{\sum_{\ell=1}^{n} c_{i\ell} k_\ell^{\beta}}$$



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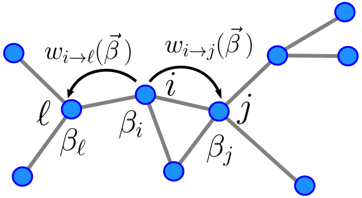

- Lazy random walks describe a dynamics where there is a constant positive probability $(1-\alpha)$ of remaining in each node

$$\bar{\mathsf{P}} = (1-\alpha)\mathsf{I} + \alpha P,\ \alpha \in (0,1)$$

We're maniacs of M-matrices

- In the standard case P is associated with L, which is an irreducible, symmetric diagonally dominant irreducible, symmetric M-matrix, $\lambda = 0$ is the lowest eigenvalue and $\omega = 1$ is the associated eigenvector.

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- In the standard case P is associated with L, which is an irreducible, symmetric diagonally dominant irreducible, symmetric M-matrix, $\lambda = 0$ is the lowest eigenvalue and $\omega = 1$ is the associated eigenvector.

- What about considering a generalized P associated with a general irreducible, symmetric M-matrix?

- The lowest eigenvalue is simple and $\lambda \geq 0$. Moreover, the associated eigenvector is positive.

All we want, all we need, is here in our Schrödinger operator

- Positive Semidefinite Schrödinger matrix

$$\boxed{\mathsf{L_q = L + D_q}}$$

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- Positive Semidefinite Schrödinger matrix

$$\mathsf{L_q} = \mathsf{L} + \mathsf{D_q}$$

- If λ is the lowest eigenvalue of $\mathsf{L_q}$ and ω is the associated eigenvector

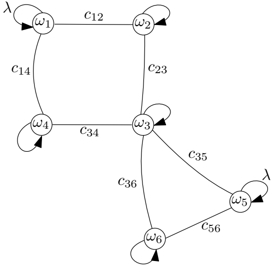
$$q = q_\omega + \lambda,\ q_\omega = -\frac{1}{\omega}\mathsf{L}\omega$$

and $\lambda \geq 0$, $\omega_i > 0$, for all $i \in V$.

All we want, all we need, is here in our Schrödinger operator

- Schrödinger Transition probability matrix

$$p_{ij}^{\lambda,\omega} = \frac{(c_{ij} + \lambda\omega_i\omega_j)\omega_j}{(k_i + q_i)\omega_i}$$



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◦ Schrödinger Transition probability matrix

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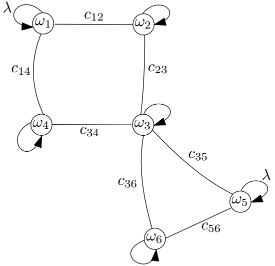

◦ There is a non-negative probability of remaining at vertex i

$$p_{ii}^{\lambda,\omega} = \frac{\lambda\omega_i^2}{k_i + q_i}$$

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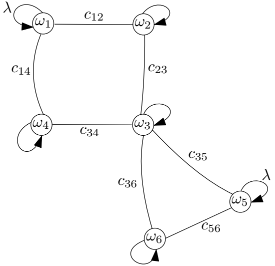

- Stationary distribution

$$\pi_i^{\lambda,\omega} = \frac{k_i + q_i\omega_i^2}{\mathsf{vol}(\Gamma)} = \frac{(k_i + q_i)\omega_i^2}{\lambda + \sum_{i,j} c_{ij}\omega_i\omega_j}$$

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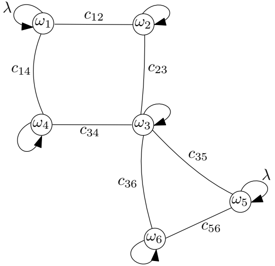

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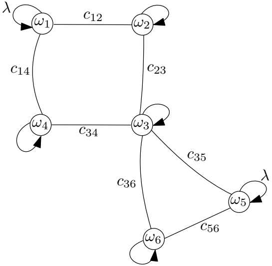
- The matrix $\mathsf{P}^{\lambda,\omega}$ is markovian and has $\pi^{\lambda,\omega}$ as stationary distribution

Oops! We do it again

- The mean first passage time matrix, $\mathsf{M}^{\lambda,\omega} = \left(m_{ij}^{\lambda,\omega}\right)_{i\neq j}$

$$p_{ij}^{\lambda,\omega} = \frac{(c_{ij} + \lambda\omega_i\omega_j)\omega_j}{(k_i + q_i)\omega_i}$$



$$\mathsf{M}^{\lambda,\omega} = \mathsf{P}^{\lambda,\omega}\mathsf{M}^{\lambda,\omega} + \mathsf{J} - \mathsf{P}^{\lambda,\omega}\mathsf{D}_{\pi^{\lambda,\omega}}^{-1} \iff \left(\mathsf{I} - \mathsf{P}^{\lambda,\omega}\right)\mathsf{M}^{\lambda,\omega} = \mathsf{J} - \mathsf{P}^{\lambda,\omega}\mathsf{D}_{\pi^{\lambda,\omega}}^{-1}$$

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$$\mathsf{M}^{\lambda,\omega} = \mathsf{P}^{\lambda,\omega}\mathsf{M}^{\lambda,\omega} + \mathsf{J} - \mathsf{P}^{\lambda,\omega}\mathsf{D}^{-1}_{\pi^{\lambda,\omega}} \iff \left(\mathsf{I} - \mathsf{P}^{\lambda,\omega}\right)\mathsf{M}^{\lambda,\omega} = \mathsf{J} - \mathsf{P}^{\lambda,\omega}\mathsf{D}^{-1}_{\pi^{\lambda,\omega}}$$

◦ Any 1-inverse or *system solving* inverse, $\mathsf{H}^{\lambda,\omega}$, provide a solution

$$\mathsf{M}^{\lambda,\omega} = \mathsf{H}^{\lambda,\omega}\left(\mathsf{J} - \mathsf{P}^{\lambda,\omega}\mathsf{D}^{-1}_{\pi^{\lambda,\omega}}\right) + \mathsf{1}\alpha^{\mathsf{T}}$$

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- Any 1-inverse or *system solving* inverse, $\mathsf{H}^{\lambda,\omega}$, provide a solution

$$\mathsf{M}^{\lambda,\omega} = \mathsf{H}^{\lambda,\omega}\big(\mathsf{J} - \mathsf{P}^{\lambda,\omega}\mathsf{D}^{-1}_{\pi^{\lambda,\omega}}\big) + 1\alpha^{\mathsf{T}}$$

- Relation between L_q and $\mathsf{I} - \mathsf{P}^{\lambda,\omega}$

$$\mathsf{I} - \mathsf{P}^{\lambda,\omega} = \mathsf{D}^{-1}_{\bar{\mathsf{k}}}\Big[\mathsf{L}_\mathsf{q} - \lambda\omega\omega^{\mathsf{T}}\Big]\mathsf{D}_\omega$$

where $\bar{\mathsf{k}}_i = (\mathsf{k}_i + \mathsf{q}_i)\omega_i$.

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where $\bar{\mathsf{k}}_i = (\mathsf{k}_i + \mathsf{q}_i)\omega_i$.

- If $\mathsf{G}^{\lambda,\omega}$ is a 1-inverse of $\mathsf{L_q} - \lambda\omega\omega^{\mathsf{T}}$ s.t. $\mathsf{G}^{\lambda,\omega}\bar{\mathsf{k}} = g\cdot\omega$, g constant

$$\mathsf{M}^{\lambda,\omega} = \mathrm{vol}(\Gamma)\mathsf{D}^{-1}_{\omega}\big(\mathsf{D}^{-1}_{\bar{\mathsf{k}}} - \mathsf{G}^{\lambda,\omega} + \mathsf{J}\mathsf{G}^{\lambda,\omega}_{d}\big)\mathsf{D}^{-1}_{\omega}$$

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◦ Relation between $\mathsf{L_q}$ and $\mathsf{I} - \mathsf{P}^{\lambda,\omega}$

$$\mathsf{I} - \mathsf{P}^{\lambda,\omega} = \mathsf{D}_{\bar{\mathsf{k}}}^{-1}\Big[\mathsf{L_q} - \lambda\omega\omega^{\mathsf{T}}\Big]\mathsf{D}_\omega$$

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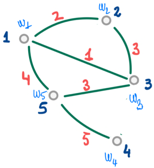
$$\mathsf{K}\left(\mathsf{M}^{\lambda,\omega}\right) = 1 - g + \mathrm{tr}\left(\mathsf{G}^{\lambda,\omega}\mathsf{D}_{\mathsf{q}+\mathsf{k}}\right)$$

A little bit of it



$\lambda = 1.458$

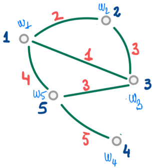
$\omega = [0.992, 0.0619, 0.0508, 0.0201, 0.0907]$

$$\boxed{\mathsf{L_q} = \mathsf{L} + \mathsf{D}_q, \quad q = q_\omega + \lambda, \quad q_\omega = -\frac{1}{\omega}\mathsf{L}\omega}$$

$q_\omega = [-6.4584, 29.5416, 21.5416, 17.5416, 34.5416]$ and $q = q_\omega + \lambda$

$$\underbrace{\begin{bmatrix} 2 & -2 & -1 & 0 & -4 \\ -2 & 36 & -3 & 0 & 0 \\ -1 & -3 & 30 & 0 & -3 \\ 0 & 0 & 0 & 24 & -5 \\ -4 & 0 & -3 & -5 & 48 \end{bmatrix}}_{\mathsf{L}_q} = \underbrace{\begin{bmatrix} 7 & -2 & -1 & 0 & -4 \\ -2 & 5 & -3 & 0 & 0 \\ -1 & -3 & 7 & 0 & -3 \\ 0 & 0 & 0 & 5 & -5 \\ -4 & 0 & -3 & -5 & 12 \end{bmatrix}}_{\mathsf{L}} + \underbrace{\begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & 31 & 0 & 0 & 0 \\ 0 & 0 & 23 & 0 & 0 \\ 0 & 0 & 0 & 19 & 0 \\ 0 & 0 & 0 & 0 & 36 \end{bmatrix}}_{\mathsf{D}_q = \mathsf{D}_{q_\omega + \lambda}}$$

A little bit of it



$\lambda = 1.458$

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- Stationary probability distribution

$$\pi^{\lambda,\omega} = \begin{bmatrix} 0.4714 \\ 0.1169 \\ 0.0651 \\ 0.0081 \\ 0.3386 \end{bmatrix} \qquad \pi = \begin{bmatrix} 0.01620 \\ 0.01157 \\ 0.01620 \\ 0.01157 \\ 0.02778 \end{bmatrix}$$

- "importance" w.r.t. λ, ω: **1, 5, 2, 3, 4**

- "importance" **1=3, 5, 2=4**

It's friday,

we're (almost) done!