Group inverses of M-matrices and applications to finite Markov chains and Laplacians of graphs

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Outline

Group Inverses and Mean First Passage Matrices

Proximity in Group Inverses of M-matrices

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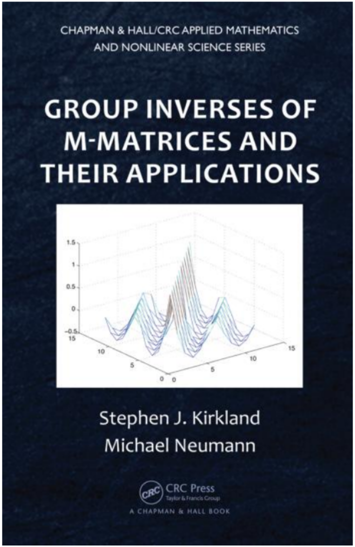
References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GROUP INVERSES OF M-MATRICES AND THEIR APPLICATIONS
Stephen J. Kirkland
Michael Neumann
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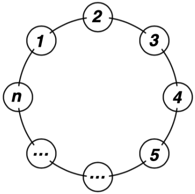
Example of a Markov chain model

Consider a ring network consisting of n nodes labeled as $1, \cdots, n$, arranged in a clockwise manner with each node connected to its (nearest) neighboring nodes by undirected edges:



This defines an *ergodic* Markov chain representing a one-dimensional periodic random walk with transition matrix

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{2} & 0 \\ 0 & 0 & \ddots & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix} \in \mathbb{R}^{n,n}.$$

The matrix T is *irreducible* and *stochastic*.

A matrix $T \in \mathbb{R}^{n,n}$ is **irreducible** if it is not permutationally similar to a matrix of the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where $A \in \mathbb{R}^{k,k}$ for some $1 \leq k < n$.

The chain is **ergodic** $\iff$ The transition matrix T is irreducible.
$\iff$ The directed graph of T is strongly connected.

Mean First Passage Matrix

Let $T \in \mathbb{R}^{n,n}$ be the transition matrix of a finite homogeneous ergodic Markov chain $\{X_k\}_{k=0}^{\infty}$ with states $\mathcal{S}_1, \cdots, \mathcal{S}_n$.

- The **first passage time** from state $\mathcal{S}_i$ to state $\mathcal{S}_j$ is the random variable

$$F_{i,j} \; = \; \min\{\ell \geq 1 : X_\ell = \mathcal{S}_j | X_0 = \mathcal{S}_i\},$$

  the smallest value of ℓ such that the chain is in state $\mathcal{S}_j$ after ℓ steps, given that the chain started in state $\mathcal{S}_i$.
- The **mean first passage time** from state $\mathcal{S}_i$ to state $\mathcal{S}_j$ is the expected value of F_{ij}:

$$m_{i,j} \; = \; \mathbb{E}(F_{i,j}) \; = \; \sum_{k=1}^{\infty} kPr(F_{i,j} \; = \; k).$$

- The matrix $M = (m_{i,j})$ is the **mean first passage matrix** of the chain.

Stationary Distribution Vector

The **stationary distribution vector** of the chain is the (positive) vector $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$ satisfying

$$\boldsymbol{\pi}^T T = \boldsymbol{\pi}^T, \quad \|\boldsymbol{\pi}\|_1 = 1.$$

That is, $\boldsymbol{\pi}$ is the **left Perron vector** of T.

For example, if T is symmetric,

$$\boldsymbol{\pi} = \left[\frac{1}{n}, \cdots, \frac{1}{n}\right]^T = \frac{1}{n}\mathbb{1}, \text{ where } \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Group Inverse

For $A \in \mathcal{R}^{n,n}$, the **group inverse**, if it exists, is the unique matrix X satisfying the matrix equations

$$AXA = A, \; XAX = X \text{ and } AX = XA.$$

If it exists, X is denoted by $A^{\#}$.

If A is nonsingular, $A^{\#} = A^{-1}$.

If A is singular, $A^{\#}$ exists $\Leftrightarrow$ 0 is a *semisimple* eigenvalue of A.

$A^{\#}$ exists $\Leftrightarrow$ rank $A =$ rank A^2.

Spectral properties of the group inverse

For singular $A \in \mathcal{R}^{n,n}$ with group inverse $A^{\#} \in \mathcal{R}^{n,n}$, the following are true:

1. $A^{\#}$ has 0 as a semisimple eigenvalue and its multiplicity coincides with the multiplicity of 0 as an eigenvalue of A.
2. A vector $\mathbf{x}$ is a right (resp. left) null vector for A if and only if it is a right (resp. left) null vector for $A^{\#}$.
3. The number $\lambda \neq 0$ is an eigenvalue of A of multiplicity m if and only if $1/\lambda$ is an eigenvalue of $A^{\#}$ of multiplicity m.
4. A vector $\mathbf{x}$ is a right (resp. left) eigenvector for A corresponding to the eigenvalue $\lambda \neq 0$ if and only if it is a right (resp. left) eigenvector for $A^{\#}$ corresponding to the eigenvalue $1/\lambda$.

M-matrices

A matrix of the form $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B)$ is called an **M–matrix**.
If $s > \rho(B)$, then $A \ = \ s\left(I - \frac{B}{s}\right) \ \Rightarrow$ (using the Neumann expansion)

$$A^{-1} = \frac{1}{s}\left(I + \frac{B}{s} + \frac{B}{s^2} + \cdots\right).$$

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Theorem. A is a non-singular M–matrix if and only if A has non-positive off-diagonal entries and $A^{-1} \geq 0$.

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Theorem. A is a non-singular M–matrix if and only if A has non-positive off-diagonal entries and $A^{-1} \geq 0$.

Theorem. If A_1 and A_2 are non-singular M–matrices with $A_1 \leq A_2$, then $A_2^{-1} \leq A_1^{-1}$.

The M-matrix $A = I - T$

If T is the transition matrix of an n-state ergodic Markov chain, then $A = I - T$ is a singular M-matrix of rank $n - 1$.

Furthermore, the group inverse $A^{\#}$ exists and $A^{\#}$ *explains virtually everything that one would want to know about the Markov chain*. (Meyer, 1975)

The group inverse associated with an M-matrix

Proposition. (Meyer 75) Let $T \in \mathcal{R}^{n,n}$ be an irreducible stochastic matrix and partition T as

$$T = \left[\begin{array}{c|c} T_{1,1} & T_{1,2} \\ \hline T_{2,1} & T_{2,2} \end{array}\right]$$

where $T_{1,1} \in \mathcal{R}^{n-1,n-1}$. Let $A = I - T$. Then $A^{\#}$ exists and can be written in partitioned form as

$$A^{\#} = (I - \mathbb{1}\boldsymbol{\pi}^T) \left[\begin{array}{c|c} (I - T_{1,1})^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] (I - \mathbb{1}\boldsymbol{\pi}^T),$$

where $\mathbb{1} = [1, \cdots, 1]^T$ and $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$ is the left Perron vector of T.

The group inverse associated with an M-matrix

Corollary. Let $T \in \mathcal{R}^{n,n}$ be an irreducible stochastic matrix and partition T as

$$T = \left[\begin{array}{c|c} T_{1,1} & T_{1,2} \\ \hline T_{2,1} & T_{2,2} \end{array}\right]$$

where $T_{1,1} \in \mathcal{R}^{n-1,n-1}$. Let $A = I - T$ and suppose that $A^{\#}$ is partitioned in conformity with the partitioning of T. Let $(A^{\#})_{1,1} \in \mathbb{R}^{n-1,n-1}$ be the leading $(n-1) \times (n-1)$ principal submatrix of $A^{\#}$. Then

$$(A^{\#})_{1,1} \;=\; A_n^{-1} + \beta W - A_n^{-1} W - W \mathcal{A}_n^{-1},$$

where $A_n = I - T_{1,1}, \;\beta = \mathbf{u}^T A_n^{-1} \mathbb{1}, \;W = \mathbb{1}\mathbf{u}^T$, and $\boldsymbol{\pi}^T = [\mathbf{u}^T \;\; \pi_n]$ is the left Perron vector of T.

Mean first passage matrix and the group inverse of an associated M-matrix

Theorem. (Meyer 75) Let $T \in \mathcal{R}^{n,n}$ be the transition matrix of an n-state ergodic Markov chain with stationary distribution vector $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$ and mean first passage matrix $M = (m_{ij})$. Let $A = I - T$. Then

$$M = (I - A^{\#} + JA_d^{\#})\Pi^{-1}$$

where

$$\begin{aligned} J &= \text{the } n \times n \text{ matrix of all } 1's, \\ A_d^{\#} &= \text{diag}(A_{1,1}^{\#}, \cdots, A_{n,n}^{\#}), \\ \Pi &= \text{diag}(\pi_1, \cdots, \pi_n). \end{aligned}$$

Mean first passage matrix and the group inverse of an associated M-matrix

The proof uses the fact that the mean first passage matrix for the transition matrix T is the unique solution M to the matrix equation

$$M = T(M - M_{\mathrm{dg}}) + J$$

where for any $n \times n$ matrix A, A_{dg} is defined as

$$A_{\mathrm{dg}} = \mathrm{diag}(a_{1,1}, \ldots, a_{n,n})$$

where $a_{1,1}, \ldots, a_{n,n}$ are the diagonal entries of A.

Observations.

1. Let $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$, the stationary distribution vector of T. Then

$$\begin{aligned}
\boldsymbol{\pi}^T M &= \boldsymbol{\pi}^T (I - A^{\#} + J A_d^{\#}) \Pi^{-1} \\
&= (\boldsymbol{\pi}^T - \underbrace{\boldsymbol{\pi}^T A^{\#}}_{\mathbf{0}^T} + \underbrace{\boldsymbol{\pi}^T J}_{\mathbb{1}^T} A_d^{\#}) \Pi^{-1} \\
&= (\boldsymbol{\pi}^T + \mathbb{1}^T \operatorname{diag}(A^{\#}_{1,1}, \ldots, A^{\#}_{n,n})) \operatorname{diag}(\tfrac{1}{\pi_1}, \ldots, \tfrac{1}{\pi_n}) \\
&= \mathbb{1}^T + \mathbb{1}^T \operatorname{diag}\left(\frac{A^{\#}_{1,1}}{\pi_1}, \ldots, \frac{A^{\#}_{n,n}}{\pi_n} \right).
\end{aligned}$$

Thus, for $k = 1, \cdots, n$,

$$(\pi^T M)_k = 1 + \frac{A^{\#}_{k,k}}{\pi_k}.$$

Observations.

2. Let $M = (m_{i,j})$ be the mean first passage matrix for the transition matrix T. Since

$$M = (I - A^{\#} + JA_d^{\#}) \begin{pmatrix} \frac{1}{\pi_1} & & \\ & \ddots & \\ & & \frac{1}{\pi_n} \end{pmatrix},$$

the (i,j)-entry of M is

$$\begin{aligned} m_{i,j} &= \textstyle\sum_{k=1}^{n} (I - A^{\#} + JA_d^{\#})_{i,k} \delta_{k,j} \frac{1}{\pi_j} \\ &= \frac{A_{j,j}^{\#} - A_{i,j}^{\#}}{\pi_j}, \text{ if } \ i \neq j, \end{aligned}$$

and

$$m_{i,i} = \frac{1}{\pi_i}.$$

Relation between the mean first passage times and row sums of inverses of principal submatrices of singular M-matrices

Let $A = I - T$, where T is the transition matrix of an n-state ergodic Markov chain. For $j = 1, \cdots, n$, let A_j be the $(n-1) \times (n-1)$ principal submatrix of A obtained by deleting its jth row and column. Then

- A_j is a nonsingular, irreducible and row diagonally dominant M–matrix.
- A_j^{-1} is a nonnegative matrix, so its row sums are nonnegative as well.

Relation between the mean first passage times and row sums of inverses of principal submatrices of singular M-matrices

Theorem. Let $A = I - T$, where T is the transition matrix of an n-state ergodic Markov chain with mean first passage matrix $M = (m_{i,j})$. For $j = 1, \cdots, n$, let

$$\bar{M}_j \;=\; [m_{1,j}, \cdots, m_{j-1,j}, m_{j+1,j}, \cdots, m_{n,j}]^T$$

(that is $\bar{M}_j$ is the j-th column of M with the j-th entry deleted). Then

$$\bar{M}_j \;=\; A_j^{-1}\mathbb{1},$$

where A_j is the $(n-1) \times (n-1)$ principal submatrix of A obtained by deleting its jth row and column, and $\mathbb{1}$ is the $(n-1)$-vector of all ones.

Relation between the mean first passage times and row sums of inverses of principal submatrices of singular M-matrices

Theorem. Let $A = I - T$, where T is the transition matrix of an n-state ergodic Markov chain with mean first passage matrix $M = (m_{i,j})$. For $j = 1, \cdots, n$, with $i \neq j$,

$$m_{ij} = (A_j^{-1}\mathbb{1})_i = \sum_{t=1}^{n-1} (A_j^{-1})_{i,t}$$

where A_j is the $(n-1) \times (n-1)$ principal submatrix of A obtained by deleting its jth row and column, and $\mathbb{1}$ is the $(n-1)$-vector of all ones.

The Kemeny Constant

Let $i \in \{1, \ldots, n\}$ be fixed and let

$$K_i = \sum_{j=1}^{n} m_{i,j}\pi_j.$$

Then K_i gives the mean first passage time from state $\mathcal{S}_i$ when the destination state is unknown.

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Theorem. [Kemeny-Snell, Hunter, ...]

$$K_i = K, \quad \text{for all } i.$$

The number K is called **Kemeny's constant.**

Kemeny's constant measures the expected number of steps from any initial state to a randomly chosen final state, and is thus regarded as an indicator of the overall transit efficiency of the corresponding Markov chain. (Xu-Kirkland 2019)

The Kemeny Constant

Theorem. $K = 1 + \text{trace}(A^{\#})$.

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Theorem. $K \; = \; 1 + \text{trace}(A^{\#})$.

Proof: Using mean first passage matrix in terms of group inverse,

$$m_{i,j} = \begin{cases} \dfrac{A^{\#}_{j,j} - A^{\#}_{i,j}}{\pi_j} & , \text{ if } \; i \neq j, \\ \dfrac{1}{\pi_j} & , \text{ if } \; i = j. \end{cases}$$

Thus,

$$\begin{aligned} K \;&=\; \sum_{j=1}^{n} m_{i,j}\pi_j \;=\; 1 + \text{trace}(A^{\#}) - A^{\#}_{i,i} - \sum_{j \neq i} A^{\#}_{i,j} \\ &=\; 1 + \text{trace}(A^{\#}) - \underbrace{\sum_{j=1}^{n} A^{\#}_{i,j}}_{0}, \end{aligned}$$

since $\mathbb{1}$ is a null vector for $A = I - T$. $\square$

The Kemeny Constant

Corollary.

$$K = 1 + \sum_{i=2}^{n} \frac{1}{1-\lambda_i},$$

where $\lambda_2, \cdots, \lambda_n$ are the eigenvalues of T other than 1.

Another formula for Kemeny constant

Theorem. For $j = 1, \cdots, n$

$$K = 1 + \mathrm{trace}(A_j^{-1}) - \frac{A^{\#}_{j,j}}{\pi_j},$$

where A_j is the $(n-1) \times (n-1)$ principal submatrix of A obtained by deleting its jth row and column.

Proof: Without loss of generality, assume $j = n$.

For $i, j = 1, \cdots, n-1,\ i \neq j$,

1. $m_{i,j}\pi_j = A^{\#}_{j,j} - A^{\#}_{i,j}$,
2. $A^{\#}_{i,j} = (A_n^{-1})_{i,j} + \beta\pi_j - (A_n^{-1}\mathbb{1})_i\pi_j - \sum_{k=1}^{n-1}\pi_k(A_n^{-1})_{k,j}$,
3. $A^{\#}_{j,j} = (A_n^{-1})_{j,j} + \beta\pi_j - (A_n^{-1}\mathbb{1})_j\pi_j - \sum_{k=1}^{n-1}\pi_k(A_n^{-1})_{k,j}$.

Proceed by performing the algebra. □

Note the observation

$$(\boldsymbol{\pi}^T M)_j \;=\; 1 + \frac{A^{\#}_{j,j}}{\pi_j}$$

implies

$$\frac{A^{\#}_{j,j}}{\pi_j} \;=\; \sum_{k \neq j} \pi_k m_{k,j}.$$

So,

$$\frac{A^{\#}_{j,j}}{\pi_j} \quad \approx$$

(weighted) average of the mean first passage times into state j from any other state.

Effect of perturbation

Question. Suppose that the matrix T of transition probabilities is perturbed by adding an error matrix E such that the resulting matrix $\overline{T} = T + E$ is still irreducible and stochastic.
How would such a perturbation affect the new Kemeny constant $\overline{K}$?

Perturbation example

Let

$$T = \begin{bmatrix} t_{1,1} & t_{1,2} & \cdots & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & \cdots & \cdots & t_{2,n} \\ \vdots & \vdots & & & \vdots \\ t_{n,1} & t_{n,2} & \cdots & \cdots & t_{n,n} \end{bmatrix}$$

and suppose that

$$\begin{aligned} \overline{T} &= T + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix} \\ &= T + \mathbb{1}\mathbf{h}^T, \quad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \ \mathbf{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \\ & \qquad \text{and } \mathbf{h}^T\mathbb{1} = 0. \end{aligned}$$

Theorem. If T and $\overline{T}$ are as above with respective Kemeny constants K and $\overline{K}$, then $\overline{K} = K$.

Proof. Follows from $K = 1 + \sum_{i=2}^{n} \frac{1}{1-\lambda_i}$,
where $\lambda_2, \cdots, \lambda_n$ are the eigenvalues of T other than 1, and the fact that T and $\overline{T}$ have the same eigenvalues.

Corollary If $\mathbf{h} = \varepsilon(\mathbf{e}_i - \mathbf{e}_j)$, then

$$\frac{\overline{A}^{\#}_{j,j}}{\overline{\pi}_j} \;\geq\; \frac{A^{\#}_{j,j}}{\pi_j},$$

and

$$\frac{\overline{A}^{\#}_{i,i}}{\overline{\pi}_i} \;\leq\; \frac{A^{\#}_{i,i}}{\pi_i}.$$

Proof. (First part) Follows from

$$\begin{aligned} K &= 1 + \text{trace}(A_j^{-1}) - \frac{A^{\#}_{j,j}}{\pi_j}, \\ \overline{K} &= 1 + \text{trace}(\overline{A}_j^{-1}) - \frac{\overline{A^{\#}}_{j,j}}{\overline{\pi}_j}, \end{aligned}$$

and the fact that A_j and $\overline{A}_j$ are non-singular M–matrices with $\overline{A}_j \leq A_j$.

Theorem Let T be a symmetric, stochastic and irreducible matrix, and suppose that $\overline{T} = T + E$, where E is a positive semi-definite matrix such that $\overline{T}$ remains stochastic and irreducible. Let $K = K(T)$ and $\overline{K} = K(\overline{T})$ be their respective Kemeny constants. Then $K \leq \overline{K}$.

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Proof: Let λ_i and $\overline{\lambda}_i$, $i = 2, \cdots, n$, be the eigenvalues of T and $\overline{T}$ respectively, that are different from 1. Then by Weyl's Theorem,

$$\lambda_i \leq \overline{\lambda}_i, \ i = 2, \cdots, n.$$

Thus,

$$\sum_{i=2}^{n} \frac{1}{1-\lambda_i} \leq \sum_{i=2}^{n} \frac{1}{1-\overline{\lambda}_i},$$

and hence, $K \leq \overline{K}$. □

Example

Consider an ergodic Markov chain representing a one-dimensional periodic random walk with transition matrix

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

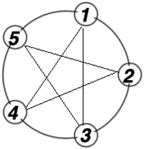
Perturb T by adding the following positive semi-definite error matrix E:

$$E = \begin{bmatrix} 0.0954 & -0.0395 & 0.0285 & 0.0127 & -0.0971 \\ -0.0395 & 0.0524 & -0.0479 & 0.0116 & 0.0234 \\ 0.0285 & -0.0479 & 0.0705 & -0.0573 & 0.0063 \\ 0.0127 & 0.0116 & -0.0573 & 0.0922 & -0.0591 \\ -0.0971 & 0.0234 & 0.0063 & -0.0591 & 0.1265 \end{bmatrix}.$$

The resulting matrix $\overline{T} = T + E$ is

$$\overline{T} = \begin{bmatrix} 0.0954 & 0.4605 & 0.0285 & 0.0127 & 0.4029 \\ 0.4605 & 0.0524 & 0.4521 & 0.0116 & 0.0234 \\ 0.0285 & 0.4521 & 0.0705 & 0.4427 & 0.0063 \\ 0.0127 & 0.0116 & 0.4427 & 0.0922 & 0.4409 \\ 0.4029 & 0.0234 & 0.0063 & 0.4409 & 0.1265 \end{bmatrix},$$

which is still irreducible and stochastic.



A perturbation of the Markov Chain
"Short–cuts" are added to non-neighboring nodes.

Then, by previous theorem, $K \leq \overline{K}$. That is, the average of the mean first passage times from any state will not decrease.

Proximity in Group Inverses of M-matrices

Let $\mathcal{G} = (V, E)$ be a connected undirected weighted graph with vertex set $V = \{1, \cdots, n\}$ and edge set E. With $\mathcal{G}$ we can associate the *Laplacian matrix* $L = (\ell_{i,j}) \in \mathbb{R}^{n,n}$, whose entries are given by

$$\ell_{i,j} = \begin{cases} -w_{i,j}, & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j \\ & \quad \text{with an edge of weight } w_{i,j} \\ 0, & \text{if } i \neq j \text{ and} \\ & \quad i \text{ is not adjacent to } j \\ -\sum_{k \neq i} \ell_{i,k}, & \text{if } i = j. \end{cases}$$

It is known that L is a symmetric, irreducible, positive semi-definite M–matrix of rank $n-1$. Hence, $L^{\#}$ exists. Also, $I + L$ is a non-singular M–matrix.

Proximity between vertices of a graph

- The problem of evaluating the *proximity* (distance) between vertices of a graph was studied by Chebotarev and Shamis [1997].
- The main vehicle is the derivation of certain properties of the matrix $Q = (I + L)^{-1} = (q_{i,j})$, where L is the Laplacian matrix of the graph.
- The entry $q_{i,j}$ is interpreted as the fraction of connectivity of the vertices i and j with respect to the total connectivity of i with all the vertices in the graph.

Theorem. (Chebotarev and Shamis [1996]): Let $Q = (I + L)^{-1} = (q_{i,j})$. Then for all $1 \leq i, j, k \leq n$,

$$q_{i.i} - q_{j,i} - q_{i,k} + q_{j,k} \; \geq \; 0. \qquad (*)$$

$(*)$ is called the **triangle inequality for proximities**.

Remark:
Let

$$\begin{aligned} d_{i,j} &= q_{i,i} + q_{j,j} - q_{i,j} - \underbrace{q_{j,i}}_{=q_{i,j}} \\ &= q_{i,i} + q_{j,j} - 2q_{i,j}. \end{aligned}$$

Then d is a *metric* on the set of vertices of $\mathcal{G}$.

We can generalize $(*)$ as follows:

Theorem Let $T \in \mathbb{R}^{n,n}$ be nonnegative, irreducible and stochastic, and let $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$, $\|\boldsymbol{\pi}\|_1 = 1$, be its normalized left Perron vector. Let $A = I - T$. Then for any $1 \leq i, j, k \leq n$,

$$\frac{A^{\#}_{i,i}}{\pi_i} - \frac{A^{\#}_{j,i}}{\pi_i} - \frac{A^{\#}_{i,k}}{\pi_k} + \frac{A^{\#}_{j,k}}{\pi_k} \ \geq \ 0.$$

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$$\frac{A^{\#}_{i,i}}{\pi_i} - \frac{A^{\#}_{j,i}}{\pi_i} - \frac{A^{\#}_{i,k}}{\pi_k} + \frac{A^{\#}_{j,k}}{\pi_k} \ \geq \ 0.$$

Proof:

(1) Triangular Inequality for Mean First Passage Times (Hunter [2002])

$$m_{i,j} \ \leq \ m_{i,k} + m_{k,j}.$$

(2) Mean first passage matrix in terms of group inverse

$$m_{i,j} = \begin{cases} \dfrac{A^{\#}_{j,j} - A^{\#}_{i,j}}{\pi_j} & , \text{ if } \ i \neq j, \\ \dfrac{1}{\pi_j} & , \text{ if } \ i = j. \end{cases}$$

Special Cases

- If T is symmetric, then $\boldsymbol{\pi} = \left[\frac{1}{n}, \cdots, \frac{1}{n}\right]^T$, i.e., $\pi_j = \dfrac{1}{n}$, for all j.
  In this case,

$$A^{\#}_{i,i} - A^{\#}_{j,i} - A^{\#}_{i,k} + A^{\#}_{j,k} \geq 0,$$

  for all $1 \leq i, j, k \leq n$,

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  $$A^{\#}_{i,i} - A^{\#}_{j,i} - A^{\#}_{i,k} + A^{\#}_{j,k} \geq 0,$$
  for all $1 \leq i, j, k \leq n$,
- For the Laplacian Matrix:
  $$L = D - W,$$
  where
  $$\begin{aligned} W &= \text{weight matrix} \\ D &= \text{diag}(d_i), \text{ where } d_i = \sum_{j \neq i} w_{i,j}. \end{aligned}$$

Let $d = \max_{1 \le i \le n} d_i > 0$. Then

$$L = dI - B,$$

where

$$B = \mathrm{diag}(d - d_i) + W \geq 0$$

and

$$\rho(B) = \|B\|_\infty = d.$$

Hence, we can write

$$L = d(I - T),$$

where $T = \frac{1}{d}B$ is stochastic and irreducible.
Thus,

$$L^{\#}_{i,i} - L^{\#}_{j,i} - L^{\#}_{i,k} + L^{\#}_{j,k} \geq 0,$$

for $1 \le i, j, k \le n$.

Generalization of CS Proximity Theorem

Theorem Let $A \in \mathbb{R}^{n,n}$ be a non-singular, irreducible, symmetric and row diagonally-dominant M–matrix. Let $B = A^{-1} = (b_{i,j})$. Then for all $1 \leq i, j, k \leq n$,

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Idea of proof:

Embed A in a singular, irreducible, symmetric M–matrix F with zero row sums:

$$F \;=\; \left[\begin{array}{c|c} A & F_{1,2} \\ \hline F_{2,1} & F_{2,2} \end{array}\right] \in \mathbb{R}^{n+1,n+1}$$

where $F_{1,2}^T = F_{2,1}$, $A\mathbb{1} + F_{1,2} = 0$, $F_{2,1}\mathbb{1} + F_{2,2} = 0$, and $\mathbb{1} \;=\; [1 \cdots 1]^T$.

Then

$$F^{\#}_{i,i} - F^{\#}_{j,i} - F^{\#}_{i,k} + F^{\#}_{j,k} \;\geq\; 0,$$

for all $1 \leq i, j, k \leq n+1$.

Let $(F^{\#})_{1,1}$ = the leading $n \times n$ principal submatrix of $F^{\#}$.
Q. How is $(F^{\#})_{1,1}$ related to A^{-1}?
Meyer [1975]:

$$\underbrace{(F^{\#})_{1,1}}_{n\times n} = A^{-1} + \tfrac{\sigma}{n+1} J - \tfrac{1}{n+1}\left(A^{-1}J + JA^{-1}\right)$$

$$= B + \tfrac{\sigma}{n+1} J - \tfrac{1}{n+1}\left(BJ + JB\right),$$

where $\sigma = \mathbb{1}^T A^{-1} \mathbb{1}/(n+1)$.
Hence, for $i, j, k = 1, \cdots, n$:

$$F^{\#}_{i,i} - F^{\#}_{j,i} - F^{\#}_{i,k} + F^{\#}_{j,k} = b_{i,i} - b_{j,i} - b_{i,k} + b_{j,k} \geq 0.$$

Applications to the Perron root function

The Perron-Frobenius Theorems.
Let $A \in \mathbb{R}^{n,n}$ be a nonnegative and irreducible matrix, and let $r = \rho(A)$. Then

1. $r > 0$ and r is a simple eigenvalue of A.
2. A has a positive eigenvector corresponding to the eigenvalue r.
3. If $0 \leq B \leq A$, then $\rho(B) \leq \rho(A)$. Equality holds $\iff A = B$.

The eigenvalue $r = \rho(A)$ is called the **Perron root** of A.

Applications to the Perron root function

Let T be a nonnegative and irreducible matrix. Then $A = \rho(T)I - T$ is a singular and irreducible $M-$matrix with a group inverse $A^{\#}$.

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For a function f of a matrix B, the notation $\frac{\partial f(B)}{\partial_{i,j}}$ denotes the partial derivative with respect to the (i,j)th entry of B.

Theorem. (Deutsch, Neumann [1984]). Let T be a nonnegative and irreducible matrix, and $A = \rho(T)I - T$. Let $\mathbf{u} = [u_1, \cdots, u_n]^T$ and $\mathbf{v} = [v_1, \cdots, v_n]^T$ be the right and left Perron vectors of T (i.e., $T\mathbf{u} = \rho(T)\mathbf{u}$ and $\mathbf{v}^T T = \rho(T)\mathbf{v}^T$), normalized so that $\mathbf{v}^T\mathbf{u} = 1$. Then

$$\frac{\partial^2 \rho(T)}{\partial_{i,j}^2} = 2 v_i u_j A^{\#}_{j,i}.$$

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$$\frac{\partial^2 \rho(T)}{\partial^2_{i,j}} = 2 v_i u_j A^{\#}_{j,i}.$$

Question. When is the Perron root a concave function of every off-diagonal entry?

The question above is equivalent to:
If A is a singular and irreducible M–matrix, when is $A^{\#}$ an M–matrix?

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If A is a singular and irreducible M–matrix, when is $A^{\#}$ an M–matrix?

Theorem. The following conditions are equivalent for $A \in \mathbb{R}^{n,n}$:
(1) A is an $M-$matrix.
(2) All the off-diagonal entries of A are nonpositive, and all of the principal minors of A are nonnegative.

Mohan, Neumann, Ramamurthy [1984] :
If A is a singular irreducible $M-$matrix, then all of the principal minors of $A^{\#}$ are nonnegative.
So, it is left to tackle the sign patterns of the off-diagonal entries of $A^{\#}$.
Question: For $i \neq j$, when is $A^{\#}_{i,j} \leq 0$?

When is $A^{\#}_{i,j} \leq 0$?

<u>**Theorem**</u> Let $T \in \mathbb{R}^{n,n}$ be nonnegative, irreducible and stochastic and let $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$, $\|\boldsymbol{\pi}\|_1 = 1$, be the normalized left Perron vector of T. Suppose that $A = I - T$ and $M = (m_{i,j})$ is the mean first passage matrix for the finite homogeneous ergodic Markov chain whose transition matrix is T. Then for any pair (i,j) with $i \neq j$ and $1 \leq i,j \leq n$,

$$A^{\#}_{i,j} \;\leq\; 0 \quad \Longleftrightarrow \quad m_{i,j} \;\geq\; \sum_{k \neq j} \pi_k m_{k,j}.$$

When is $A^{\#}_{i,j} \leq 0$?

Theorem Let $T \in \mathbb{R}^{n,n}$ be nonnegative, irreducible and stochastic and let $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$, $\|\boldsymbol{\pi}\|_1 = 1$, be the normalized left Perron vector of T. Suppose that $A = I - T$ and $M = (m_{i,j})$ is the mean first passage matrix for the finite homogeneous ergodic Markov chain whose transition matrix is T. Then for any pair (i, j) with $i \neq j$ and $1 \leq i, j \leq n$,

$$A^{\#}_{i,j} \leq 0 \iff m_{i,j} \geq \sum_{k \neq j} \pi_k m_{k,j}.$$

Proof:

$$A^{\#}_{i,j} \leq 0 \iff m_{i,j} \geq \frac{A^{\#}_{j,j} - A^{\#}_{i,j}}{\pi_j} \geq \frac{A^{\#}_{j,j}}{\pi_j}.$$

When is $A^{\#}_{i,j} \leq 0$?

We can use the two different representations for mean first passage times to derive a correspondence involving the maximum row sum of the inverse of a deleted submatrix A_j and the minimum entry in the jth column of the group inverse of A.

When is $A^{\#}_{i,j} \leq 0$?

Theorem Let $T \in \mathbb{R}^{n,n}$ be nonnegative, irreducible and stochastic and let $\boldsymbol{\pi} = [\pi_1, \cdots, \pi_n]^T$, $\|\boldsymbol{\pi}\|_1 = 1$, be the normalized left Perron vector of T. Let $A = I - T$. For each $j = 1, \cdots, n$, let A_j be the $(n-1) \times (n-1)$ principal submatrix of A obtained by deleting the j–th row and column of A. For each $\ell = 1, \cdots, n-1$, let

$$r_\ell^{(j)} = \sum_{t=1}^{n-1} (A_j^{-1})_{\ell,t}. \quad (1)$$

(That is, $r_\ell^{(j)} = A_j^{-1}\mathbb{1}$, the ℓ–th row sum of A_j^{-1}.)

Let $1 \leq i \leq n$ with $i \neq j$. Then $A^{\#}_{i,j} \leq 0$ if and only if

$$r_k^{(j)} \geq \sum_{s=1}^{j-1} \pi_s r_s^{(j)} + \sum_{s=j}^{n-1} \pi_{j+1} r_s^{(j)}, \quad (2)$$

where

$$k = \begin{cases} i, & \text{if } 1 \leq i < j \\ i-1, & \text{if } j < i \leq n. \end{cases} \quad (3)$$

In particular, suppose that $1 \leq k \leq n-1$ is an index for which

$$\sum_{\nu=1}^{n-1} (A_j^{-1})_{k,\nu} \;=\; \max_{1\leq \nu \leq n-1} \sum_{\nu=1}^{n-1} (A_j^{-1})_{\mu,\nu} \;=\; \|A_j^{-1}\|_\infty .$$

Then $A^{\#}_{p,j} < 0$, where

$$p \;=\; \begin{cases} k, & \text{if } 1 \leq k < j \\ k+1, & \text{if } j \leq k \leq n-1. \end{cases} \tag{4}$$

Examples

Consider an ergodic Markov chain representing a one-dimensional periodic random walk with transition matrix

$$T = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix}.$$

and let

$$A = I - T = \begin{bmatrix} 1 & -0.5 & 0 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 & 0 \\ 0 & -0.5 & 1 & -0.5 & 0 \\ 0 & 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & 0 & -0.5 & 1 \end{bmatrix}.$$

Then the group inverse $A^{\#}$ of A is given by

$$A^{\#} = \begin{bmatrix} 0.8 & 0 & -0.4 & -0.4 & 0 \\ 0 & 0.8 & 0 & -0.4 & -0.4 \\ -0.4 & 0 & 0.8 & 0 & -0.4 \\ -0.4 & -0.4 & 0 & 0.8 & 0 \\ 0 & -0.4 & -0.4 & 0 & 0.8 \end{bmatrix}.$$

Examples

The group inverse $A^{\#}$

$$A^{\#} = \begin{bmatrix} 0.8 & 0 & -0.4 & -0.4 & 0 \\ 0 & 0.8 & 0 & -0.4 & -0.4 \\ -0.4 & 0 & 0.8 & 0 & -0.4 \\ -0.4 & -0.4 & 0 & 0.8 & 0 \\ 0 & -0.4 & -0.4 & 0 & 0.8 \end{bmatrix}$$

is an M-matrix.

The mean first passage matrix for the finite ergodic homogeneous Markov chain with transition matrix T is

$$M = \begin{bmatrix} 5 & 4 & 6 & 6 & 4 \\ 4 & 5 & 4 & 6 & 6 \\ 6 & 4 & 5 & 4 & 6 \\ 6 & 6 & 4 & 5 & 4 \\ 4 & 6 & 6 & 4 & 5 \end{bmatrix}.$$

Note that the right side of the condition

$$A^{\#}_{i,j} \leq 0 \iff m_{i,j} \geq \sum_{k \neq j} \pi_k m_{k,j}.$$

is in this case equivalent to: $m_{i,j} \geq \frac{1}{5} \sum_{k \neq j} m_{k,j}.$

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Thank You!